STiCM

Select / Special Topics in Classical Mechanics

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STiCM Lecture 35

Unit 11 : **Chaotic Dynamical Systems**

Unit 11

Chaotic Dynamical Systems

Complex behavior of simple systems!

"I am convinced that chaos research will bring about a revolution in natural sciences similar to that produced by quantum mechanics". -Gerd Binnig,

-Nobel Prize (1986) for designing Scanning Tunneling Microscope

Many others, who work in a wide variety of frontier research fields have expressed a similar view.

Physics addresses the temporal-evolution of the ‘state of a system’.

That’s what an equation of motion

(Newton / Lagrange / Hamilton / Schrodinger)

is about!

Growth of science:

Empirical knowledge, theoretical models,

predictions, testing ……

Observations of natural phenomena – Galileo / Raman ……

What laws of nature can we learn from Mathematics?

–From numbers,

for example: $\pi, \; e, \; . \; . \; . \; .$

or, from a sequence of numbers.....

Fibonnacci (1202): How many pairs of rabbits can there be if they breed in "ideal" conditions and never die?

Our rabbits **never die.**

The female always produces one new pair every month.

New pair: always

one male and one female.

How many pairs will there be in one year?

http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibnat.html#rabeecow
21/10/2010

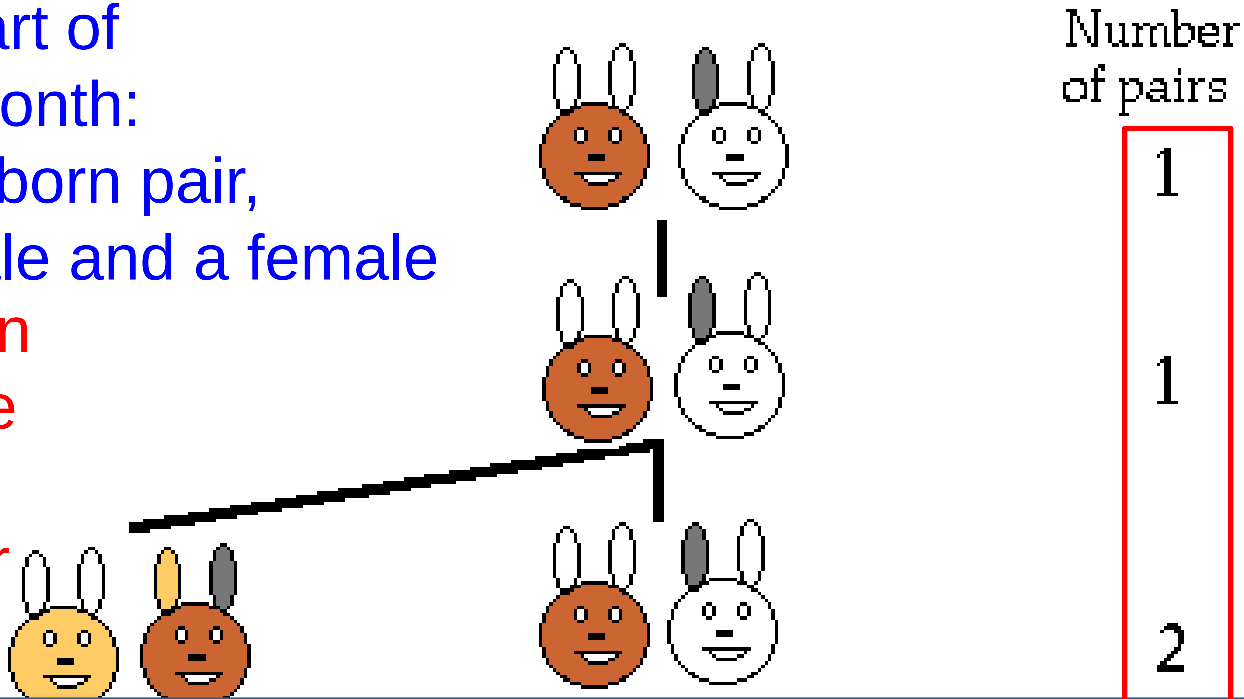
Each pair will reproduce; none will die.

Each new born pair takes a month to mature enough to mate.

The female then takes a month to deliver the next pair – always a male and a female

@start of
1st month:
new born pair,
a male and a female

Each new born pair would take 2 months to deliver another pair: M+F.



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for example: $\pi, \ e, \ . \ . \ . \ .$

or, from a sequence of numbers.....

1,1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144,...

1,2,3,5,8,13,21,34,55,89,144,…..

$2/1 = 2$

$3/2 = 1.5$

$5/3 = 1.66666...$

$8/5 = 1.6$

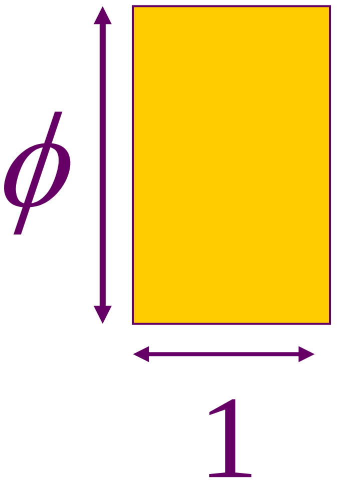
the golden ratio

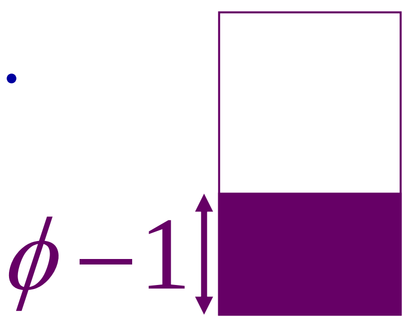
$13/8 = 1.625$

$21/13 = 1.615384...$

$34/21 = 1.61904...$

$55/34 = 1.617646...$

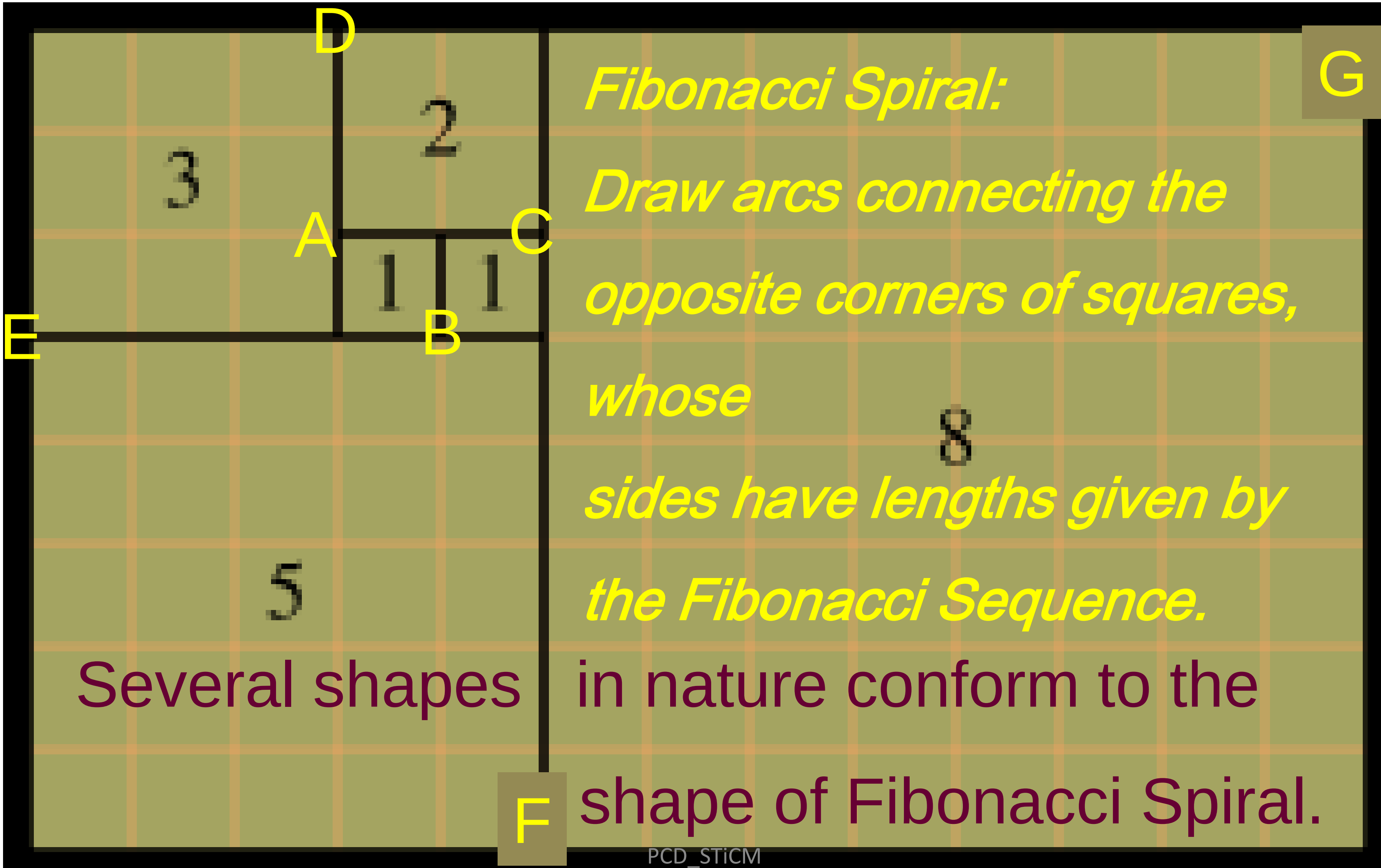




$$\phi = 1.6180339887\ldots$$

1,2,3,5,8,13,21,34,55,89,144,…..

the golden ratio = 1.6180339887…



Fibonacci Spiral:

Draw arcs connecting the opposite corners of squares, whose sides have lengths given by the Fibonacci Sequence.

Several shapes in nature conform to the shape of Fibonacci Spiral.

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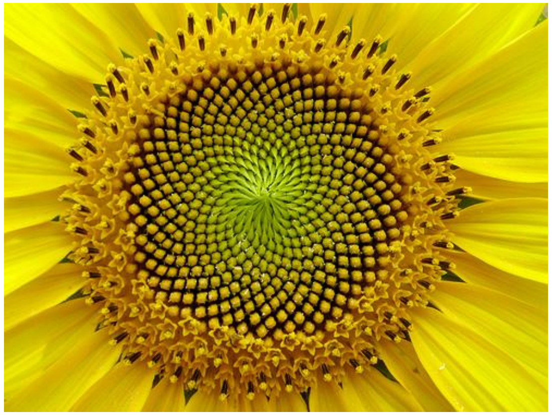
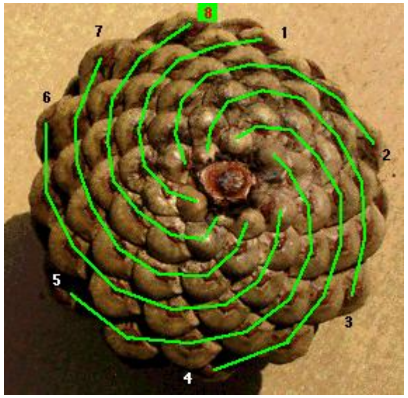
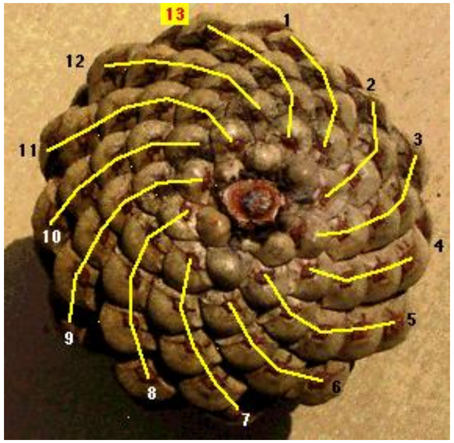
1,2,3,5,8,13,21,34,55,89,144,…..

the golden ratio = 1.6180339887…

http://hynesva.com/blogs/character_and_excellence/archive/2009/11/15/the-golden-ratio-a-wonder-of-god-s-creation.aspx







D E Knuth '*The Art of Computer Programming: Volume 1'*

(errata to second edition): "Before Fibonacci wrote his work, the sequence **F(n)** had already been discussed by Indian scholars, who had long been interested in rhythmic patterns that are formed from one-beat and two-beat notes.
The number of such rhythms having **n** beats altogether is **F(n+1)**; therefore both Gopala (before 1135) and Hemachandra (c. 1150) mentioned the numbers 1, 2, 3, 5, 8, 13, 21, ... explicitly".

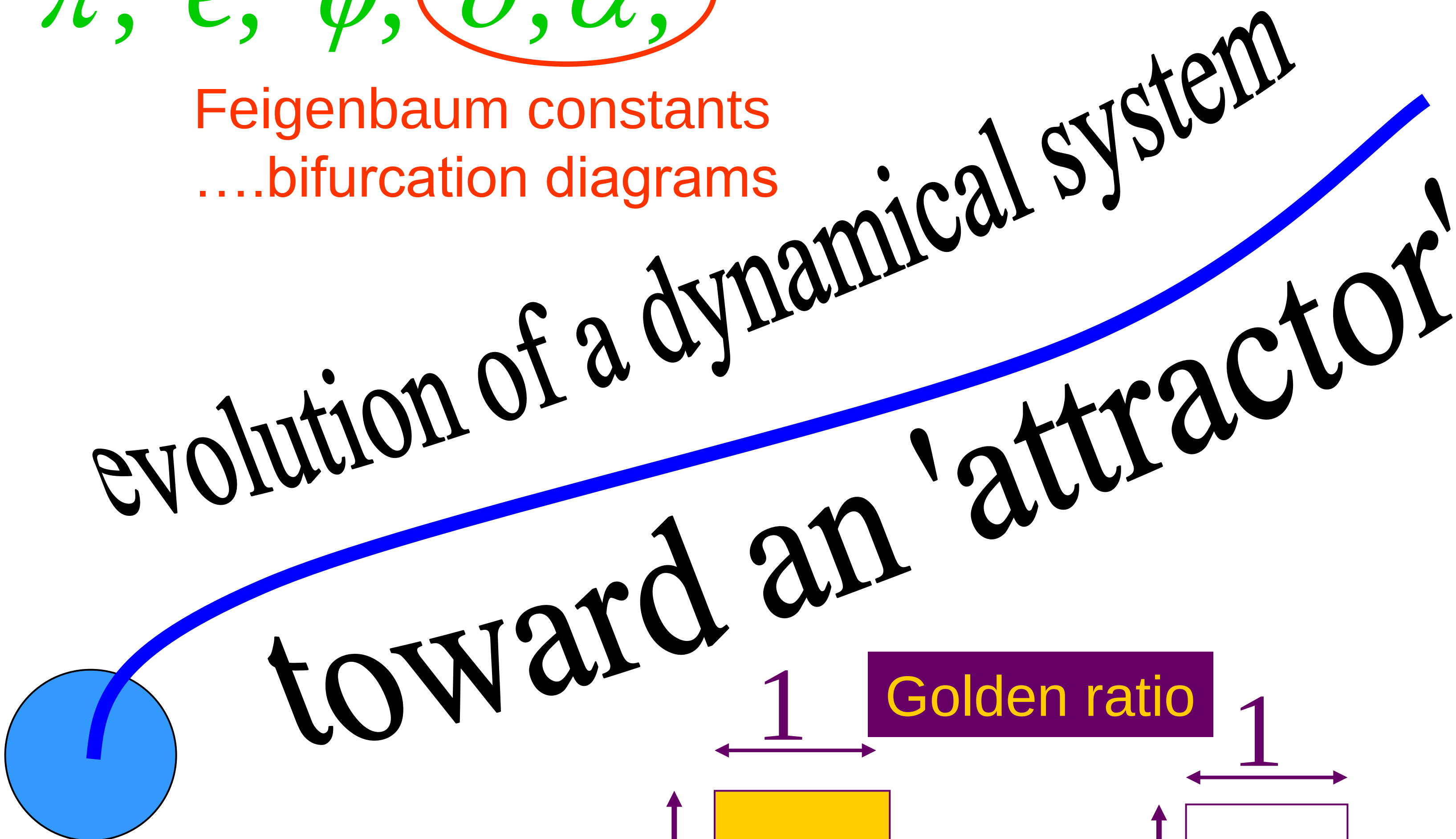
http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibBio.html

Fibunacci: Leonardo of Pisa, 'Liber Abaci' (1202)
-- but was this sequence knon earlier?

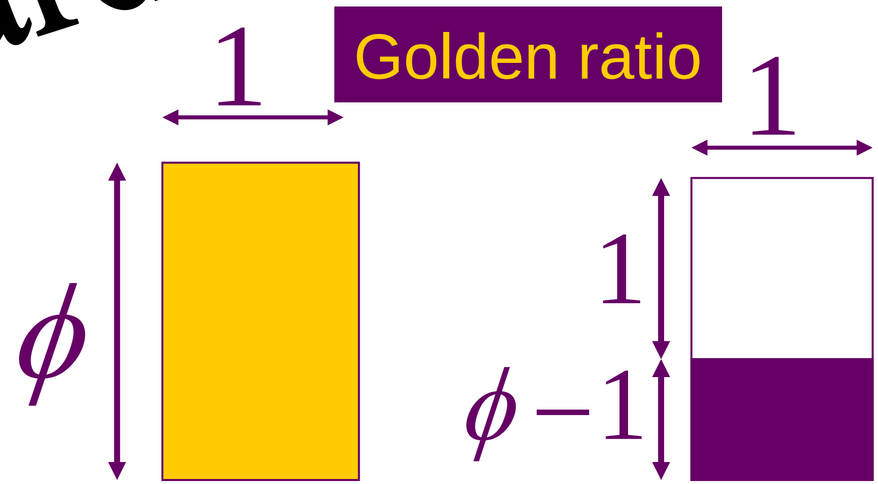
$\pi, \ e, \ \phi, \delta, \alpha,$

Feigenbaum constants

….bifurcation diagrams

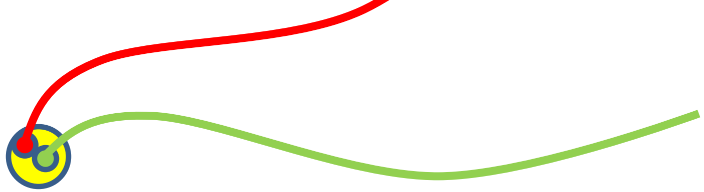


INITIAL STATE



the golden ratio = 1.6180339887…

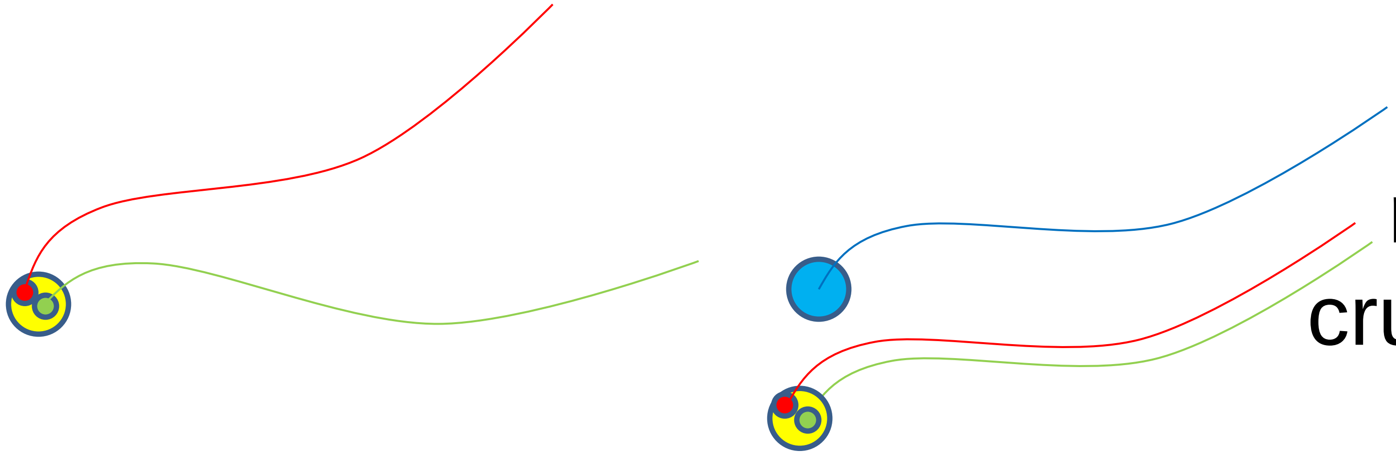
…..it may happen that small differences in the initial conditions produce very great ones in the final phenomena….



evolution of a dynamical system

*K*olmogorov, *A*rnold and *M*oser

Henri Poincaré (1854 -1912)



number crunching

6.87	10.7	10.66
+3.79	+ 9.89	+9.89
10.66	20.59	20.55

"For want of a nail,

the shoe was lost;

For want of a shoe,

the horse was lost;

For want of the horse,

the rider was lost;

For want of a rider,

the battle was lost;

For want of a battle,

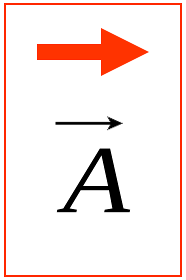
the kingdom was lost!"

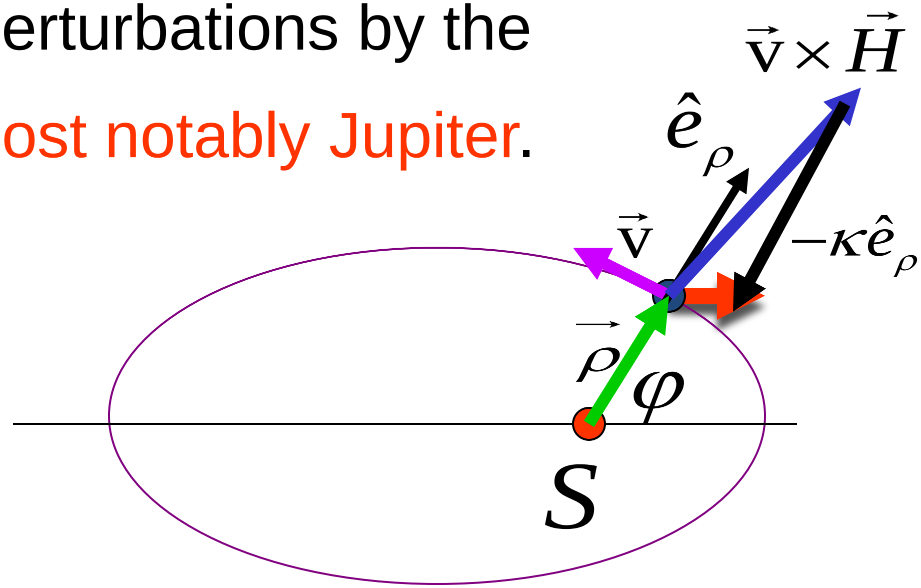
James Gleick's book on 'Chaos'

page 23 (1998 Edition)

Laplace Runge Lenz

Earth's elliptic orbit precesses, at a current rate of 0.3 degrees per century due to perturbations by the other planets, most notably Jupiter.





$$\vec{A} = \left(\vec{v} \times \vec{H}\right) - \kappa \hat{e}_\rho$$

The $\odot \vec{H}$ (specific) angular momentum vector is *out* of the plane of this figure, *toward us*.

Jacques Laskar (1989, Paris) - numerical integration of the Solar System over 200 million years.

-averaged equations, had some 150,000 terms.

Laskar's work: Earth's orbit → chaotic.

(*as well as the orbits of all the inner planets*)

An error as small as 15 meters in measuring the position of the Earth today would make it impossible to predict where the Earth would be in over 100 million years' time.

See 'Solar system dynamics' by Murray & Dermott

Dynamical System: “dynamical” : changing....

study of temporal evolution of systems/processes.

Examples:

Weather – changes with time

Changes in Chemicals – as reactions take place....

Population changes....

Motion of simple pendulum

Stock market....

..... Physics / Chemistry / Engineering / Finance / Biology ....

Question:

Can we make accurate long-time predictions?

Dynamical Systems

Newton/Lagrange/Hamilton

1890s: Poincare

1920-60: Borkhoff
Kolmogorov
Arnol'd
Moser
} *KAM*

1963: Lorenz
1970s: Ruelle & Takens
May
Feigenbaum
Mandelbrot

1980s+
Cascading of interest and work in non-linear dynamics, chaos, fractals

Our interest:

Is the evolution of a system/process predictable?

"Unpredictability"

Chaos: Even if number of variables is just one,

- and even if there is no quantum phenomenon

For example: Add 2 to the previous number, beginning with 0

0+2=2

2+2=4 examine the predictability of

4+2=6 the results of successive iterations......

6+2=8 ..... and so on

Put Rs 1000 in the bank at 10% annual interest.

A0=1000

A1=A0+0.1A0=1000+100=1100=1.1A0 N=10; Rs2593.74

A2=A1+0.1A1=1100+110=1210=1.1A1

A3=A2+0.1A2=1210+121=1331=1.1A2 N=50; Rs1,17,390.85

.... $A_N=1.1A_{N-1}=(1.1)^N(A0)$

Thomas R. Malthus (1798):
mathematical model of population growth.

Exponential growth model:

Each member of a population reproduces at the same per-capita rate, the growth rate is

r : fecundity
-ability to reproduce
-rate coefficient
-'control' parameter

$$\frac{dN}{dt} = rN$$

$$\frac{dN}{N} = rdt$$

$$log_e N = rt + c$$

$$At \text{ t=0, } log_e N(at \ t = 0) = c; \text{ i.e., } c = \log_e N_0$$

$$log_e N = rt + \log_e N_0$$

$$N(t) = e^{rt + \log_e N_0} = e^{rt} e^{\log_e N_0} = N_0 e^{rt}$$

We shall take a break here…….

Questions ?

Comments ?

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http://www.physics.iitm.ac.in/~labs/amp/

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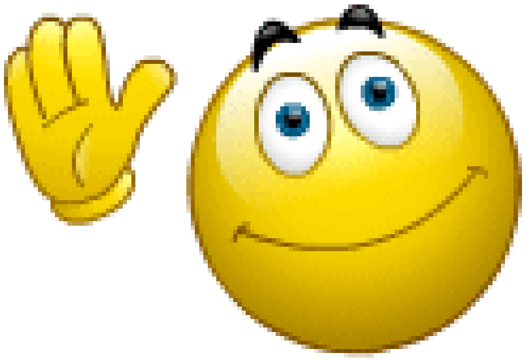

Next: L36

Unit 11 – CHAOTIC DYNAMICAL SYSTEMS

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STiCM Lecture 36

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- bifurcations, chaos!

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mathematical model of population growth.

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$$N(t) = e^{rt + \log_e N_0} = e^{rt} e^{\log_e N_0} = N_0 e^{rt}$$

Malthus's population model predicts population growth without bound for r > 0, or certain extinction for r<0.

$$N(t) = N_0 e^{rt}$$

‘Logistic’ Population Model

Two parameters:

r: growth rate.

K: carrying capacity of the system.

Carrying Capacity: population level at which the birth and death rates of a species precisely match, resulting in a stable population over time.

$$\frac{dN}{dt} = rN$$

Malthus

(exponential)

Logistic Model of Population Growth Rate / incorporates a 'feedback mechanism'

Pierre Verhulst (Belgian, 1838): the rate of population increase may be limited, depending on 'population'.

$$\frac{dN}{dt} == \left[r\left(1 - \frac{N}{K}\right) \right] N = rN\left(1 - \frac{N}{K}\right)$$

K: "carrying capacity"; N: population size.

The growth rate decreases as population size increases.

$$\frac{dN}{dt} = \left[r\left(1 - \frac{N}{K}\right) \right] N$$

This non-linear equation is known as LOGISTIC EQUATION.

$$when \ \frac{dN}{dt} = \dot{N} \geq 0$$

$$and \ the \ growth \ rate \ coefficient \ r \rangle 0,$$

$$we \ have: \quad 0 \leq N \leq K$$

$$\dot{N} = 0 \text{ when } N = 0 \ or \text{ when } N = K$$

$$N = 0 \ and \ N = K \ are \ the \ equilibrium \ values \ of \ N.$$

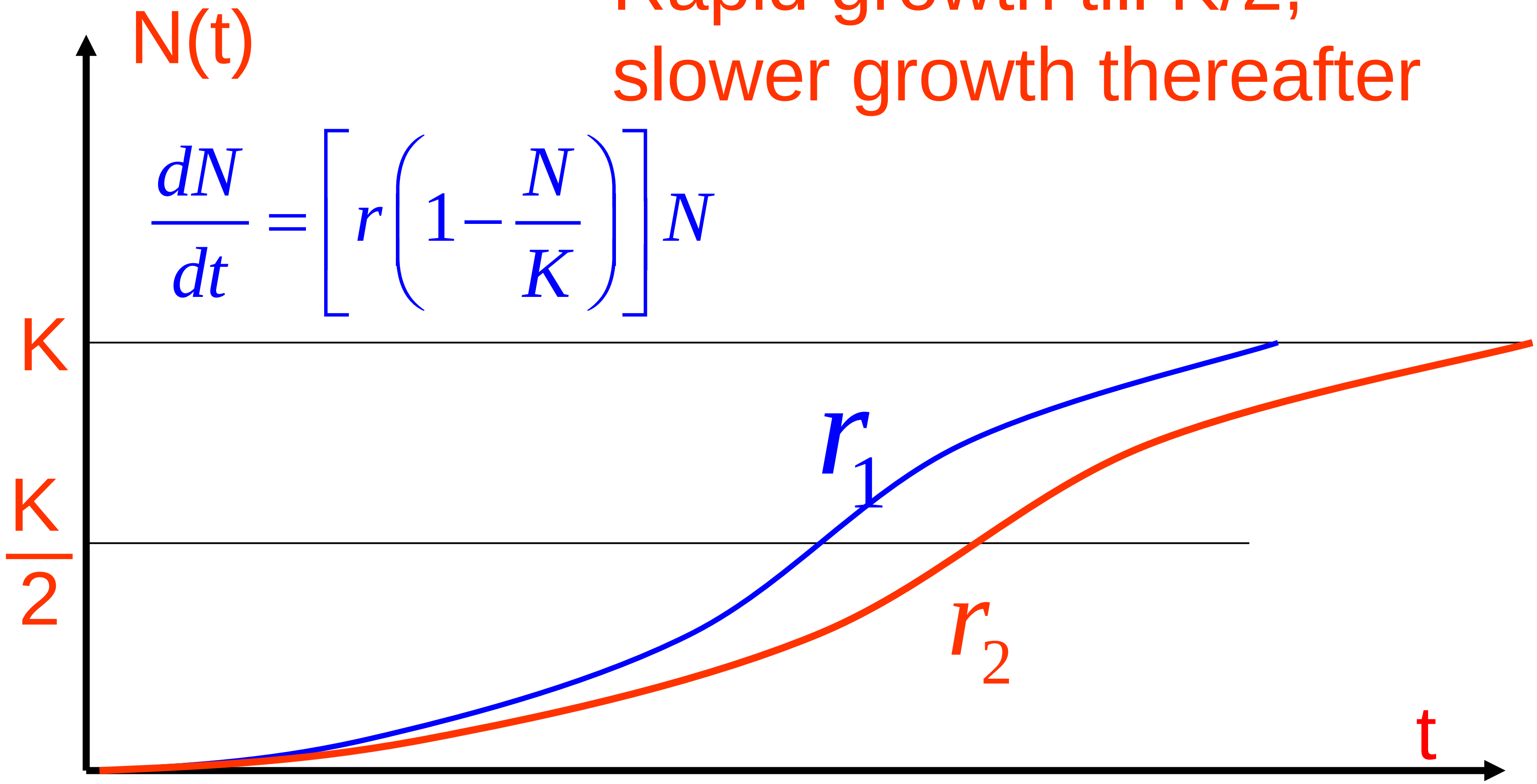
Over a passage of time,
N moves toward K.

Thus: N=0: Unstable state
N=K: Asymptotically Stable.

The LOGISTIC non-linear differential equation (continuous changes) does not predict any chaos.

Rapid growth till K/2,
slower growth thereafter

$$\frac{dN}{dt} = \left[r\left(1 - \frac{N}{K}\right)\right] N$$



K/2 is an inflection point

$$r_1 \langle \; r_2$$

$\dot{N} = 0$ when $N = 0$ *or* $N = K$

$N = 0$ *and* $N = K$ *are the equilibrium values of* N.

Reproduction: considered to be continuous in time.
N(t): continuous, analytical function of time.

Several organisms reproduce in discrete intervals.

“**How Many Pairs of Rabbits Are Created by One Pair in One Year**?” - Fibonacci

$$\frac{dN}{dt} = \left[r\left(1 - \frac{N}{K}\right) \right] N$$

← LOGISTIC, non-linear differential equation

is not applicable for 'discrete' growth models

$$\frac{N((n+1)\delta t) - N(n\delta t)}{\delta t} = r\left[1 - \frac{N(n\delta t)}{K}\right] N(n\delta t).$$

Note the correspondence, considering the very definition

$$\frac{dN}{dt} = \lim_{\delta t \to 0} \frac{\delta N}{\delta t}$$

$$\frac{dN}{dt} = rN$$

Malthus

(exponential)

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$$\frac{N((n+1)\delta t)-N(n\delta t)}{\delta t}=rN(n\delta t)\left[1-\frac{N(n\delta t)}{K}\right].$$

$$N((n+1)\delta t)-N(n\delta t)=rN(n\delta t)\left[1-\frac{N(n\delta t)}{K}\right]\delta t$$

$$N((n+1)\delta t)=N(n\delta t)+rN(n\delta t)\left[1-\frac{N(n\delta t)}{K}\right]\delta t$$

Pierre Francois Verhulst

(28/10/1804–15/2/1849 Belgium)

$$\frac{dN}{dt} = \left[r\left(1 - \frac{N}{K}\right) \right] N$$

Robert M. May
born 8 January 1936

"I urge that people be introduced to the logistic equation early in their mathematics equation."
– Robert M. May
'Simple mathematical models with very complicated dynamics'
NATURE 261 (1976) p459-467

$$P_{n+1} = rP_n(1 - P_n)$$

$n : n^{th}$ *generation index*

Logistic MAP, Difference Equation

The discrete model

$$N((n+1)\delta t) = N(n\delta t) + r\left[1 - \frac{N(n\delta t)}{K}\right] N(n\delta t)\delta t$$

gives results that are very different from those obtained from the continuum model*!*

$$\frac{dN}{dt} = \left[r\left(1 - \frac{N}{K}\right)\right] N$$

The continuum model gives the rest state $N = K$ as asymptotically stable,

- regardless of the value of r,

whereas,

the discrete model is very sensitive to the growth rate as well as the interval length between reproduction.

For large enough $r\delta t$ *, predictions of the discrete model can give rise to instabilities!*

Behavior: bizarre, chaotic!

MAP: Time domain is discrete; discrete time intervals: difference equations instead of differential equations

- **Population:**

$$P_{next} = F(P_{current})$$

linear function

$$P_{next} = rP_{current} \quad \text{(Malthus)} \rightarrow \text{linear}$$

$$P_{n+1} = rP_n(1 - P_n)$$

The modification through $(1 - P_n)$

checks the growth,

since $(1 - P_n)$ decreases as P_n increases.

The non-linear term plays havoc!

Let us see what the non-linear term does -

– depending on the value of the control parameter

- **Population:**

$$P_{next} = F(P_{current})$$

linear function

$$P_{next} = rP_{current} \text{ (Malthus)} \rightarrow \text{linear}$$

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The modification through $(1 - P_n)$

checks the growth,

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Let $r = 2.7$ (*arbitrary value – example* from James Gleick's book: Chaos - making a new science)

Starting population: $P_0 = 0.02$

$1 - 0.02 = 0.98$

$2.7 \times 0.02 \times 0.98 = 0.0529$

population $\rangle$ doubled!

next :

$2.7 \times 0.0529 \times (1 - 0.0529)$

$= 2.7 \times 0.0529 \times 0.9471 = 0.1353$

$P_{n+1} = rP_n(1-P_n)$ Logistic MAP Difference Equation

Let $r = 2.7$

Starting population: $P_0 = 0.02$

$1 - 0.02 = 0.98$

$2.7 \times 0.02 \times 0.98 = 0.0529$

Note: population has more than doubled.

next :

$2.7 \times 0.0529 \times (1 - 0.0529)$

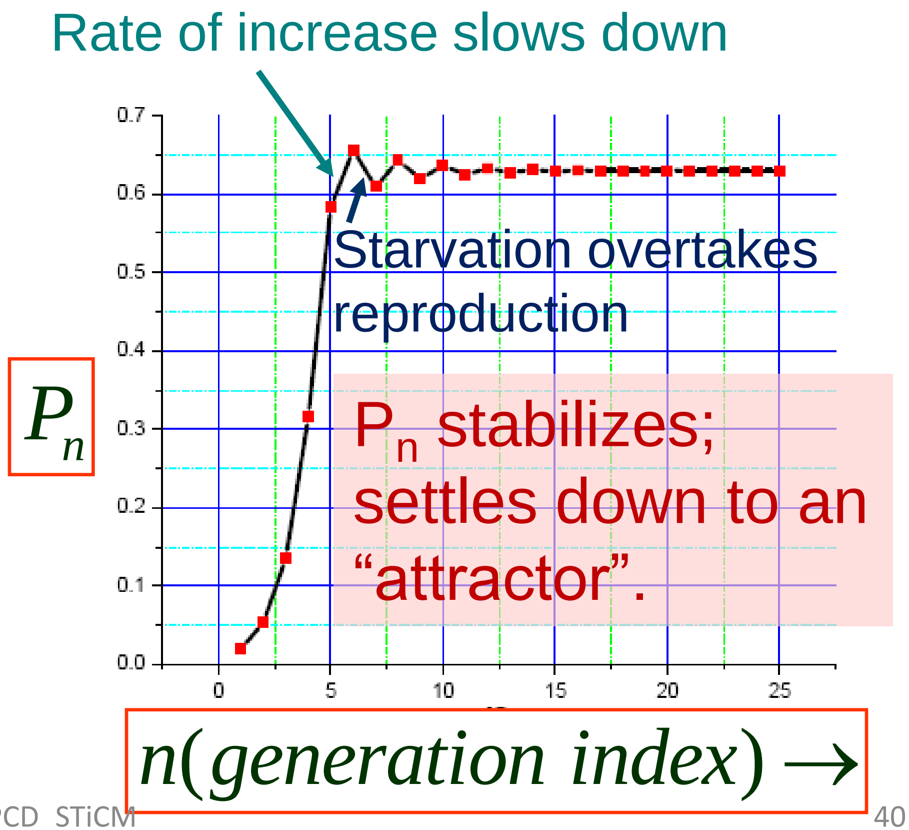
$= 2.7 \times 0.0529 \times 0.9471 = 0.1353$

next :

$2.7 \times 0.1353 \times (1 - 0.1353) = 0.3159$

1	0.02
2	0.0529
3	0.1353
4	0.3159
5	0.5835
6	0.6562
7	0.6092
8	0.6428
9	0.6199
10	0.6362
11	0.6249
12	0.6328

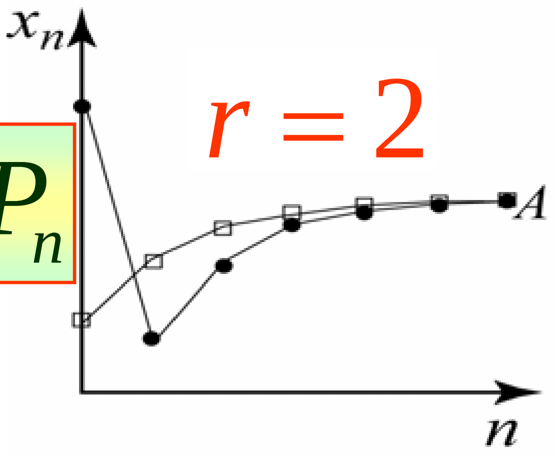
13	0.6273
14	0.6312
15	0.6285
16	0.6304
17	0.6291
18	0.63
19	0.6294
20	0.6299
21	0.6295
22	0.6297
23	0.6296
24	0.6296
25	0.6296



An ‘attractor’ is a region in the configuration or phase space that is invariant under time evolution and attracts nearby configurations -

– *those that lie within the ‘basin of attractors’.*

P_n



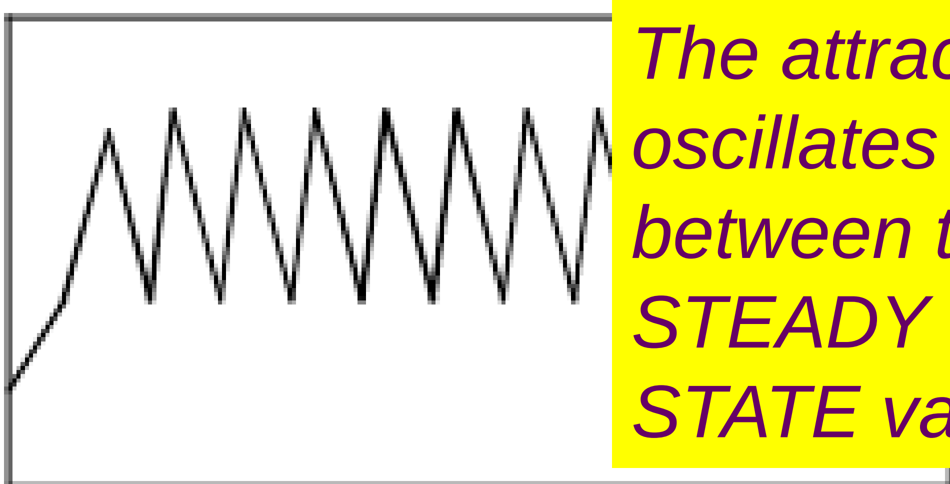
$r = 2$

Period Two Oscillations

$$\begin{cases} P_n = 0.56 & 1 \\ P_{n+1} = 0.76 & 2 \\ P_{n+2} = 0.56 & 1 \\ P_{n+3} = 0.76 & 2 \end{cases}$$

….so on

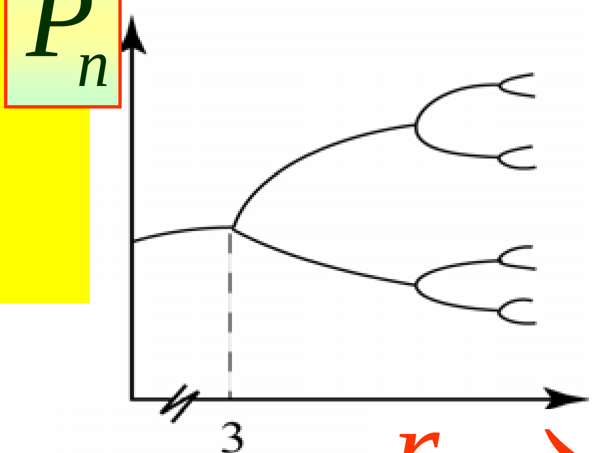
$r = 3.1$



The attractor oscillates between two STEADY STATE values

Number of iterations →

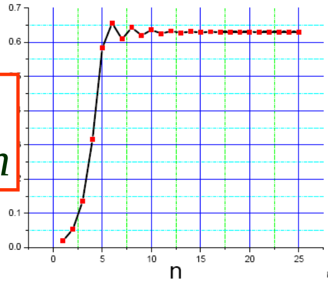
P_n



$r \rightarrow$

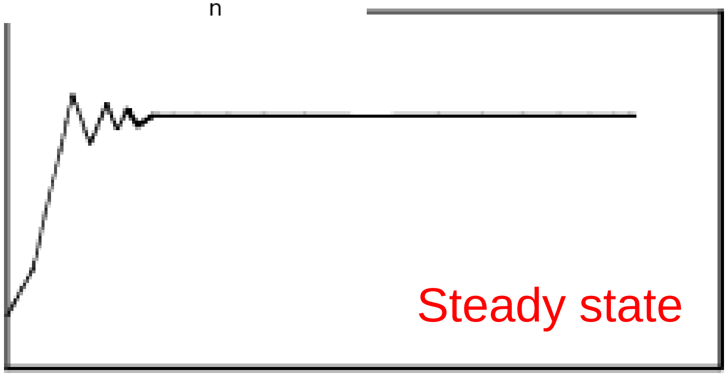
Period Doubling / Bifurcation $P_{n+1} = rP_n(1-P_n)$

P_n

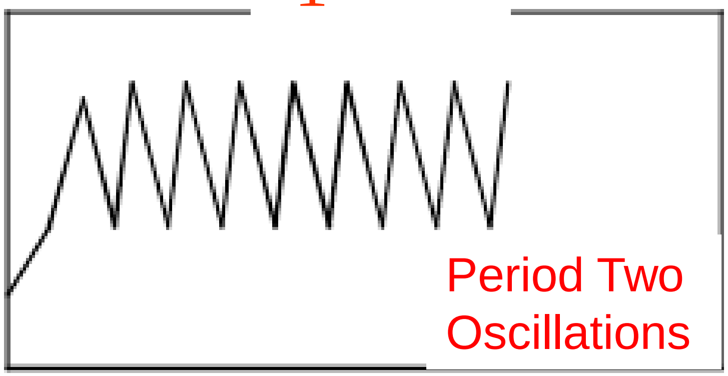


2.7

$r_1 = 3$



$P_{n+1} =$
$rP_n(1-P_n)$

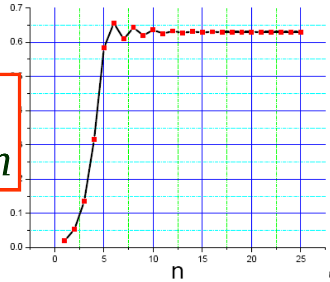


Number of iterations of the equation ⟶

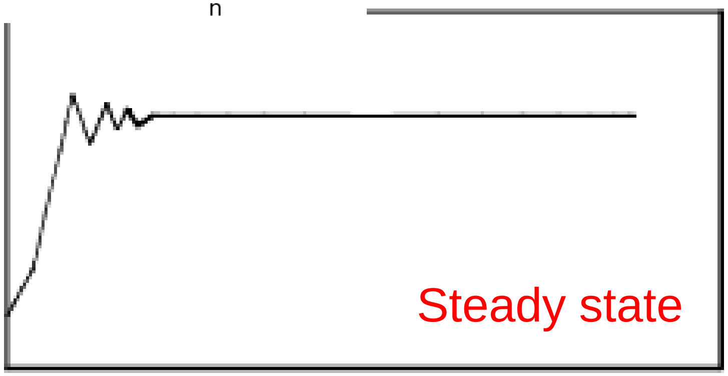
Period Doubling / Bifurcation

$$P_{n+1} = rP_n(1-P_n)$$

P_n

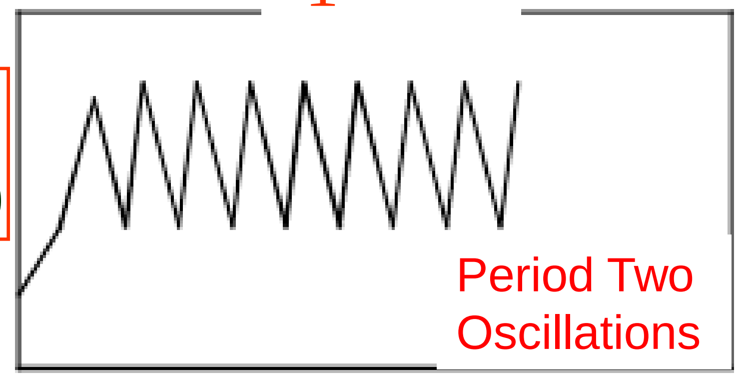


2.7



$$P_{n+1} = rP_n(1-P_n)$$

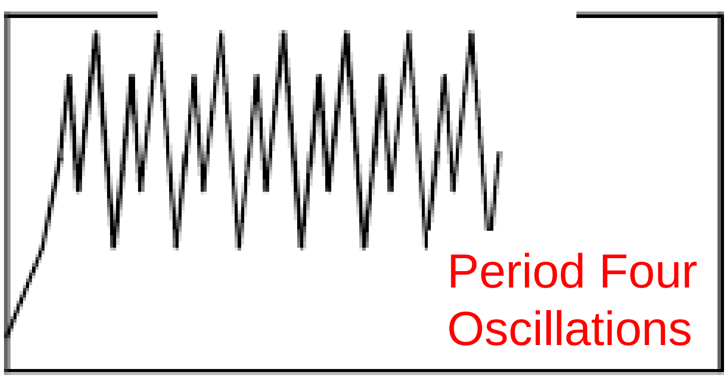
$r_1 = 3$



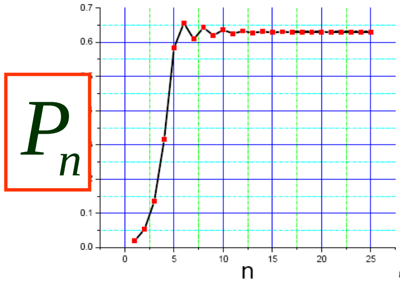
Number of iterations of the equation ⟶

$$n(generation\ index) \rightarrow$$

$r_2 \approx 3.45$



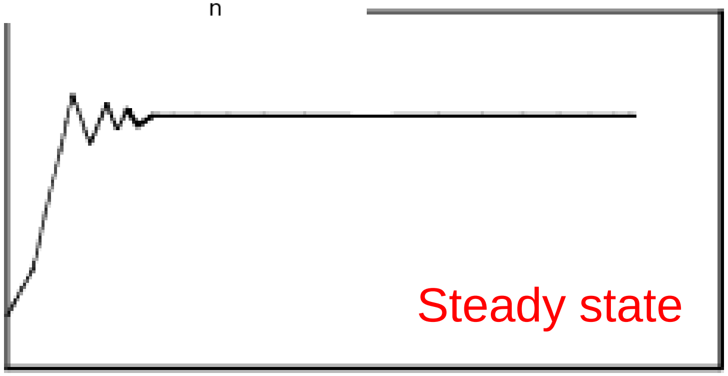
Number of iterations of the equation ⟶



Period Doubling / Bifurcation

$$P_{n+1} = rP_n(1-P_n)$$

2.7



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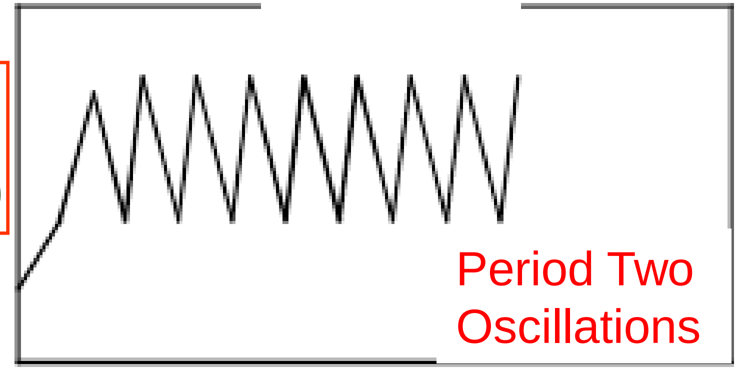

Number of iterations of the equation ⟶

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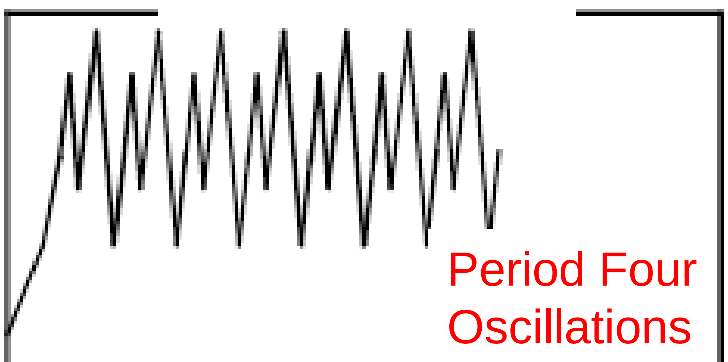


$$P_{n+1} = rP_n(1-P_n)$$

$$r \rangle 3.57$$

Number of iterations of the equation ⟶

An **attractor** is a set to which a dynamical system evolves over a long enough time.

That is, points that get close enough to the attractor remain close even if slightly disturbed.

An ‘attractor’ can be a point, a curve, a manifold, or even a complicated set with a fractal structure known as a *strange attractor.*

CHAOS theory: builds mathematically rigorous formulations to describe the ‘attractors’ of chaotic dynamical systems.

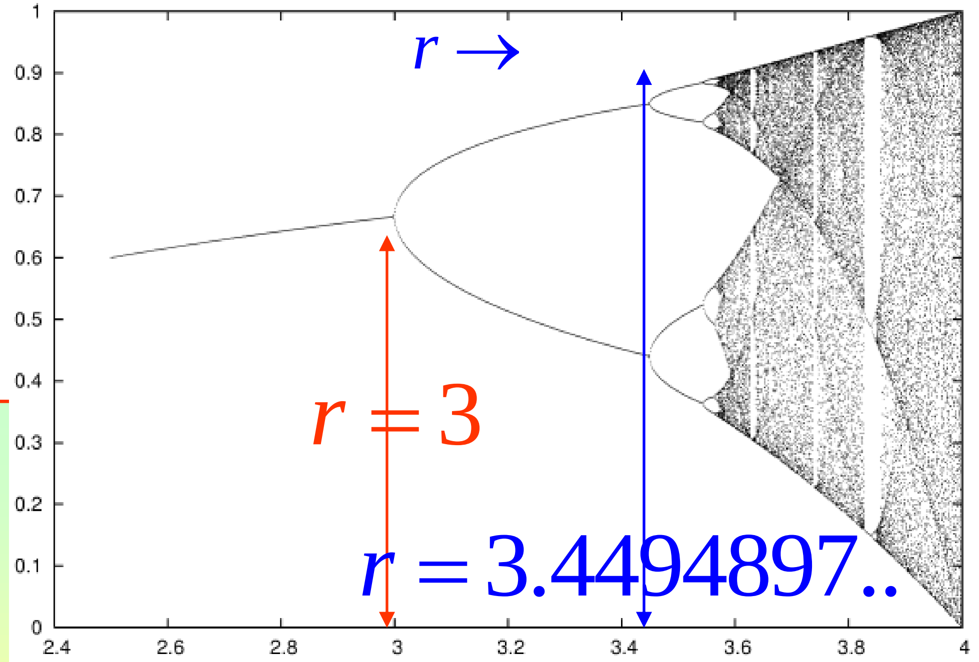
r fecundity

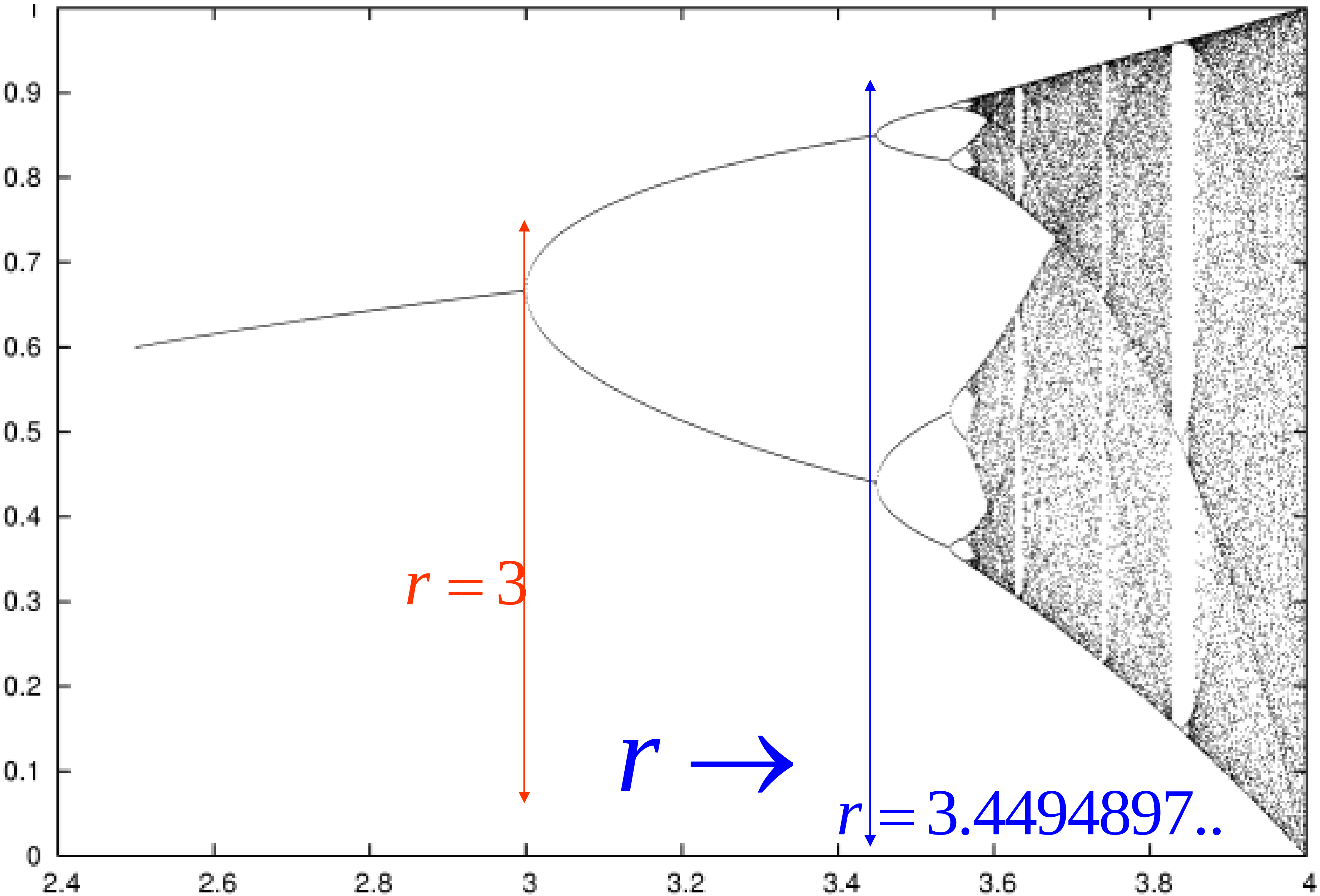
-ability to reproduce

-control parameter

$$P_{next} = rP_{current}(1 - P_{current})$$

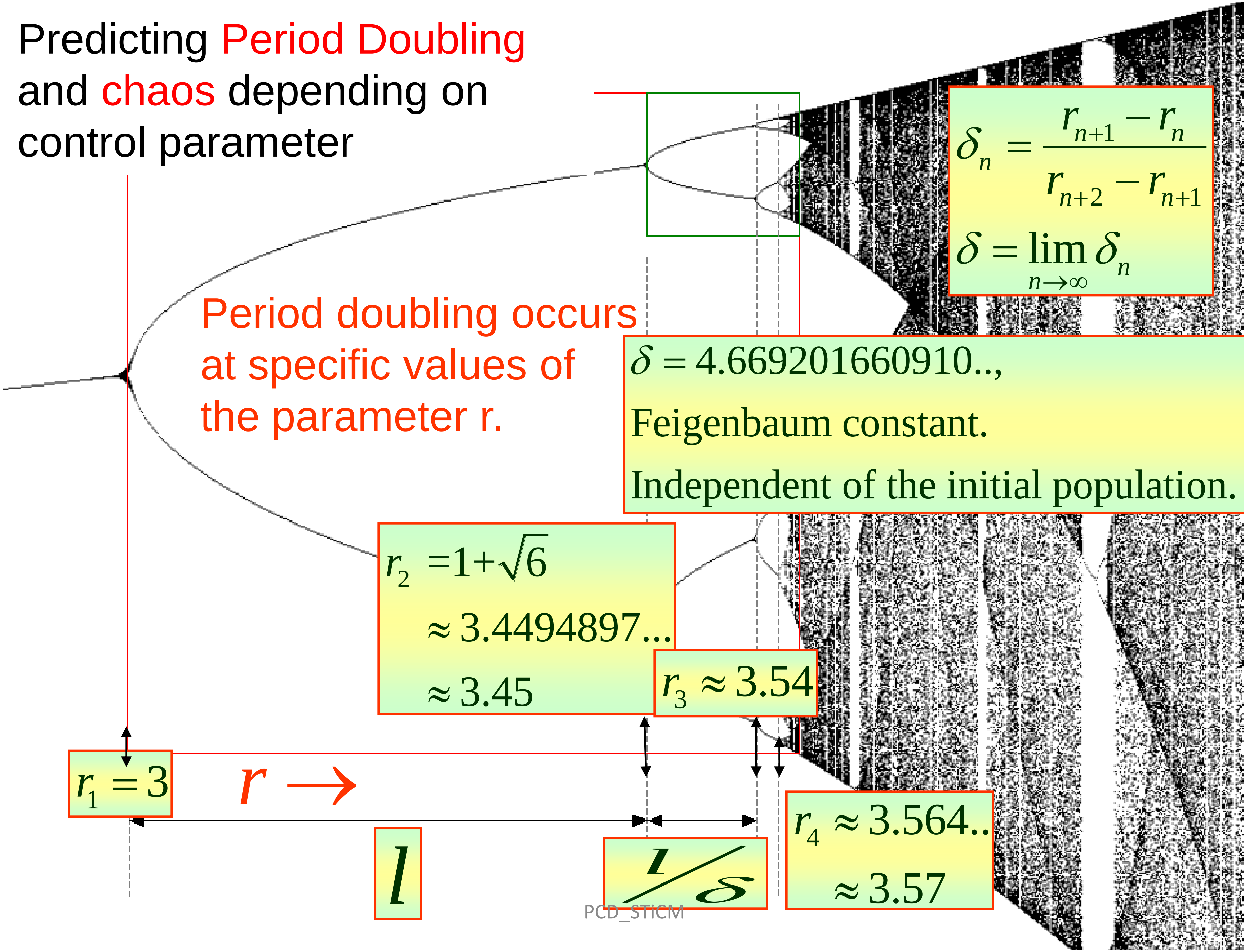
$0 < r < 1$: Population eventually dies, no matter what the initial population.





1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
2.4
2.6
2.8
3
3.2
3.4
3.6
3.8
4
$r = 3$
$r \rightarrow$
$r = 3.4494897..$

Predicting Period Doubling and chaos depending on control parameter



Period doubling occurs at specific values of the parameter r.

$$\delta_n = \frac{r_{n+1} - r_n}{r_{n+2} - r_{n+1}}$$

$$\delta = \lim_{n \to \infty} \delta_n$$

$\delta = 4.669201660910..,$

Feigenbaum constant.

Independent of the initial population.

$r_1 = 3$

$r_2 = 1 + \sqrt{6}$

$\approx 3.4494897...$

≈ 3.45

$r_3 \approx 3.54$

$r_4 \approx 3.564..$

≈ 3.57

We shall take a break here…….

Questions ?

Comments ?

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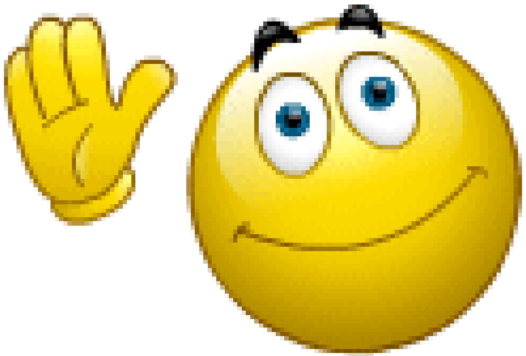
http://www.physics.iitm.ac.in/~labs/amp/

pcdeshmukh@iitmandi.ac.in



Next: L37

Unit 11 – CHAOTIC DYNAMICAL SYSTEMS

STiCM

Select / Special Topics in Classical Mechanics

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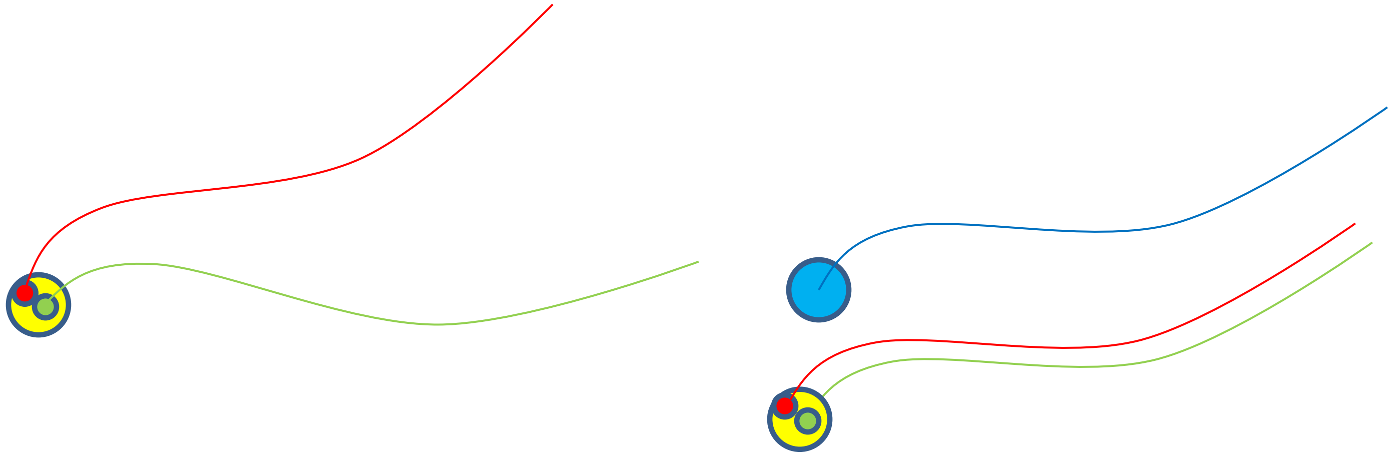
STiCM Lecture 37

Unit 11 : **Chaotic Dynamical Systems**

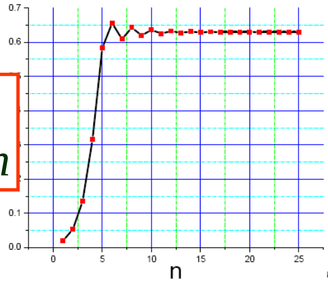
- Bifurcations, Chaos! 'Attractor', 'Strange Attractor'



…..butterfly effect

Period Doubling / Bifurcation $P_{n+1} = rP_n(1-P_n)$

P_n



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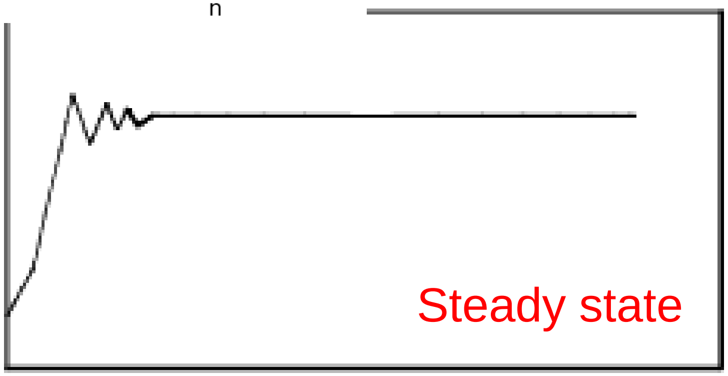

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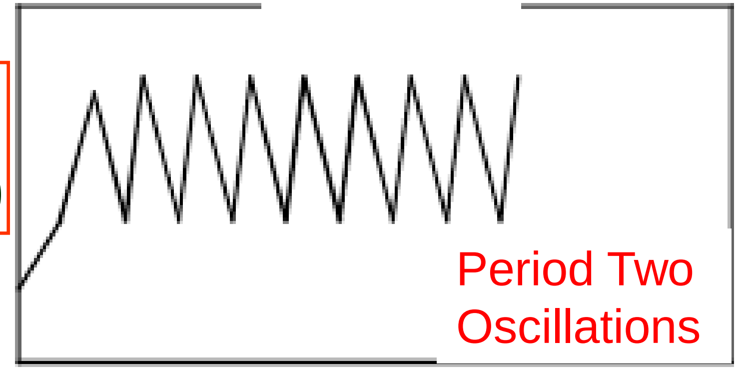

Number of iterations of the equation ⟶

$$n(generation\ index) \rightarrow$$

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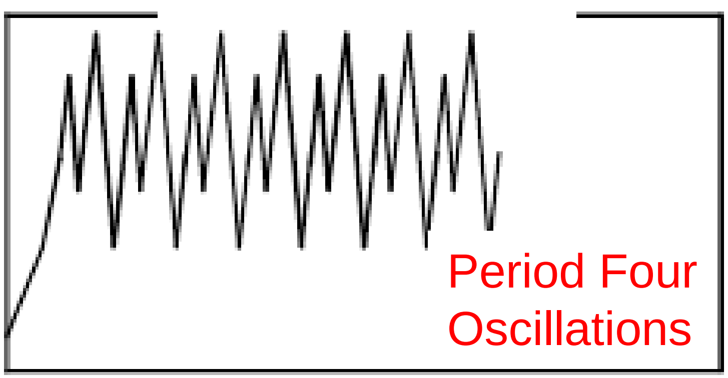
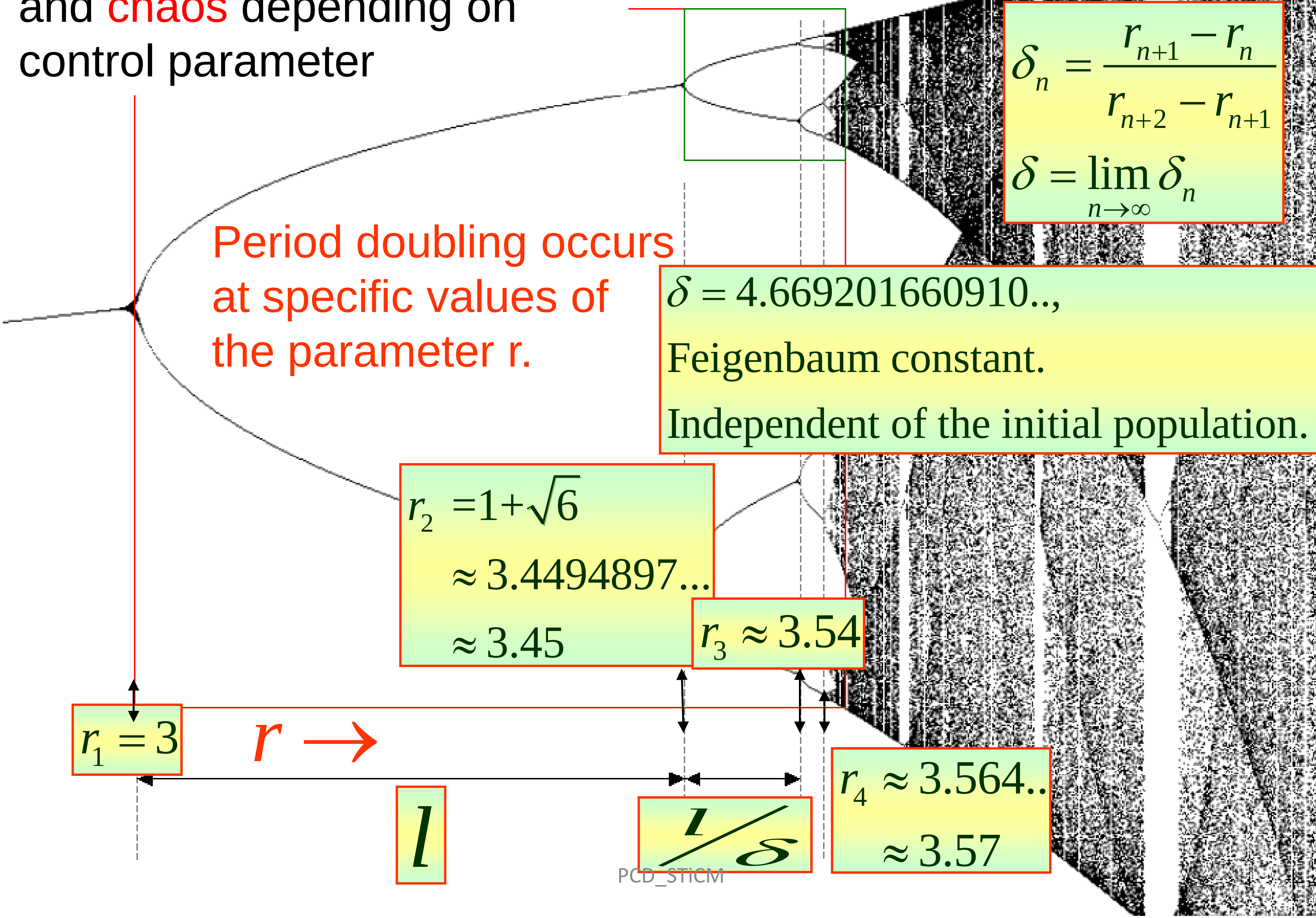


$$P_{n+1} = rP_n(1-P_n)$$

$$r \rangle 3.57$$

Number of iterations of the equation ⟶

Predicting Period Doubling and chaos depending on control parameter



Chaotic ≠ Random

Random: same initial value may result in unpredictable final state.

Chaotic: deterministic.

Same initial value results in same final state, but the final state is very sensitive to small variations in the initial value.

Since initial values cannot be known with infinite accuracy, the outcome can be chaotic/unpredictable: butterfly effect

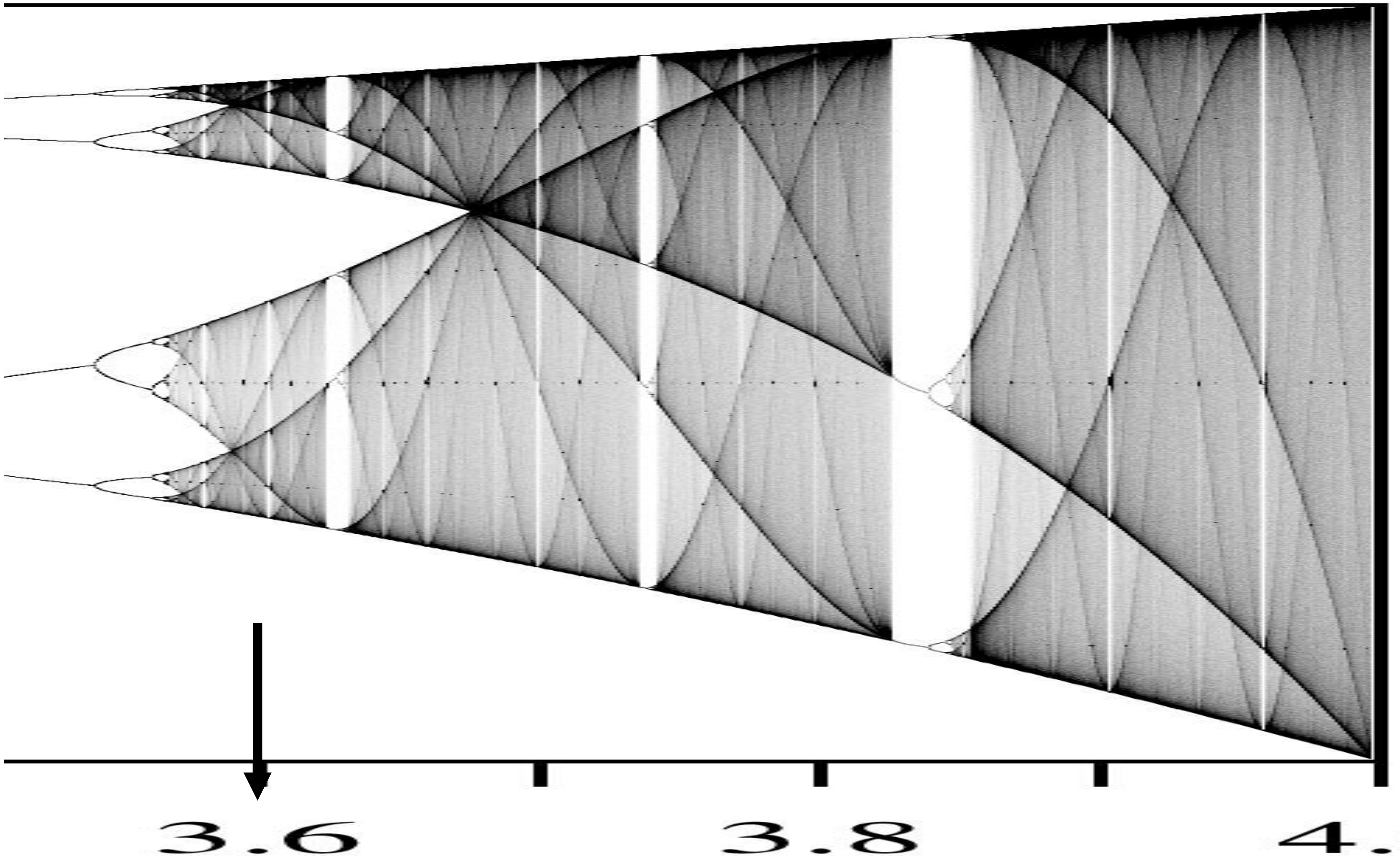
Mitchell Jay Feigenbaum (b. Dec. 19, 1944)

Feigenbaum's constant can be used to predict when chaos will occur.

When the value of the driving parameter *r* equals 3.57, P_{next} neither converges nor oscillates — its value becomes completely random!

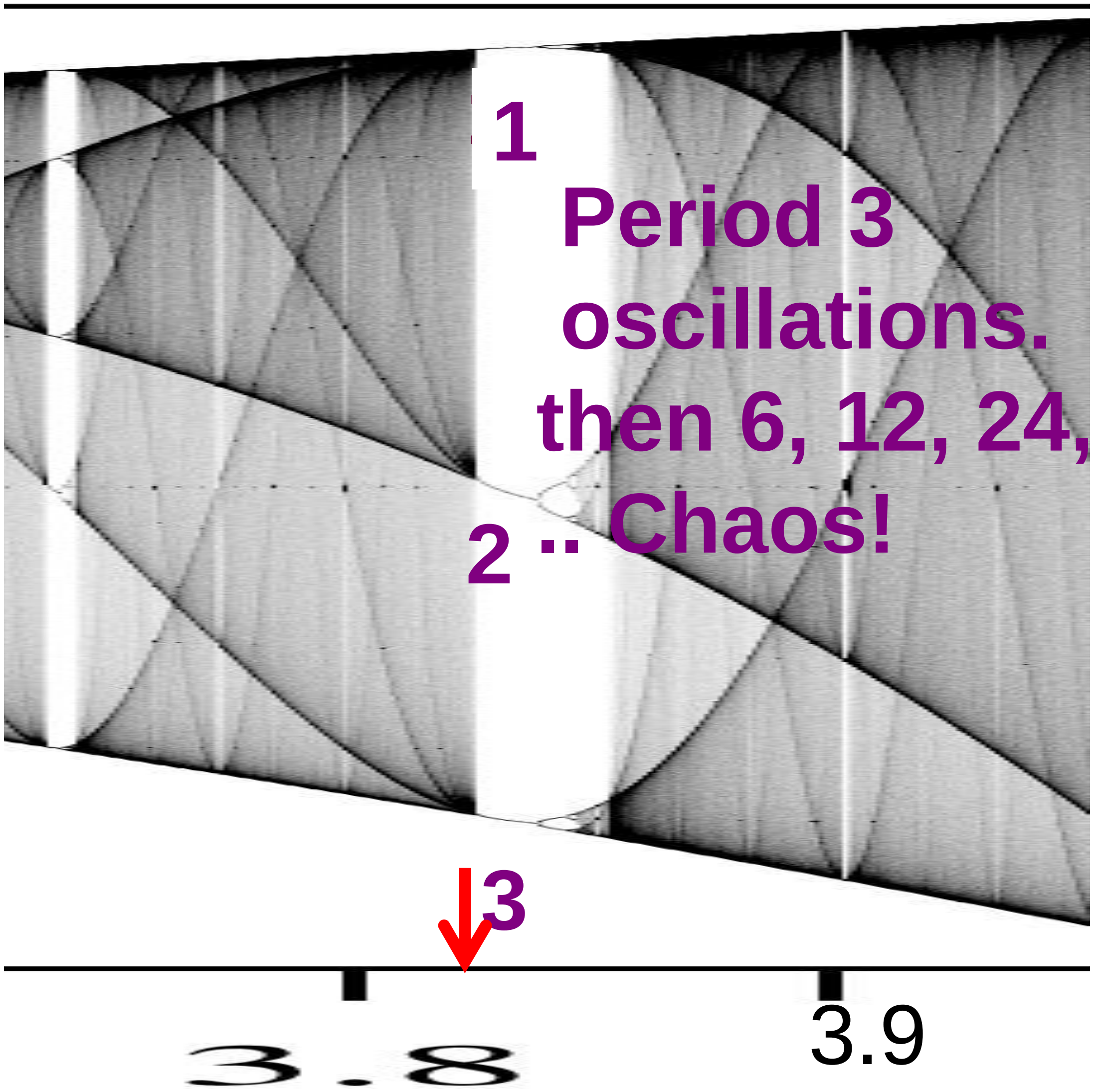
$$r_4 \approx 3.564.. \approx 3.57$$

For values of *r* larger than 3.57, the behavior is mostly chaotic.



For most values of r>3.57 : chaotic behavior.

For certain isolated values of *r*, *we see* non-chaotic behavior.



In any one-dimensional system, if a regular cycle of period three ever appears, then the system will display regular cycles of every other length, as well as completely chaotic cycles.

"PERIOD THREE IMPLIES CHAOS".

– James Yorke

$$P_{next} = rP_{current}(1 - P_{current})$$

We have an in-built non-linearity in the above relation

$$\ddot{x} = -\frac{k}{m}x \rightarrow linear$$

$$\ddot{\theta} = -\frac{g}{l}\sin\theta \rightarrow non-linear$$

$$linearization : \sin\theta \approx \theta$$

For a non-linear system,
the principle of linear superposition will not hold. *OF COURSE!*

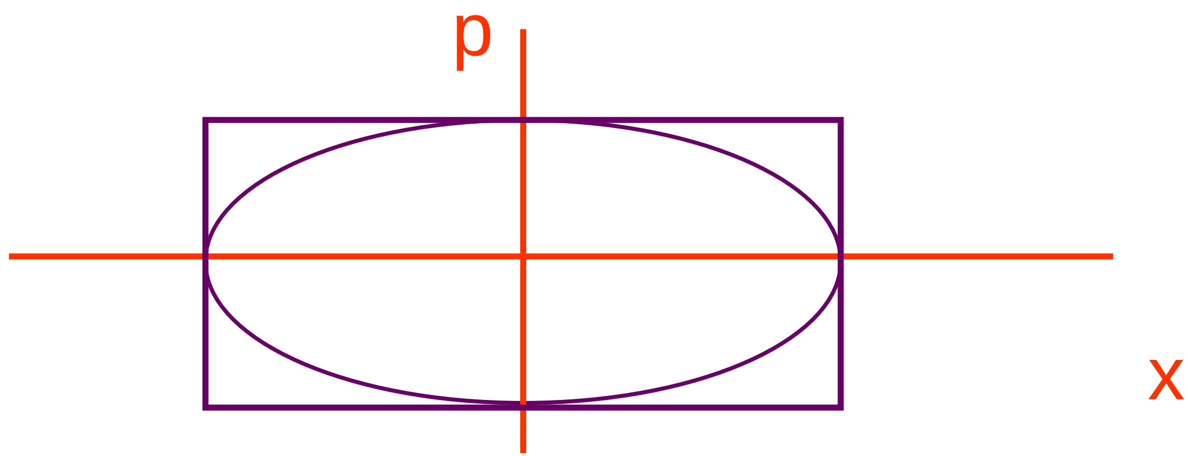
Linear systems are easier to treat since parts of the system can be separated, solved independently, and the solutions superposed to get the answer.

For a non-linear system, one cannot do this!

$$\frac{1}{2}kx^2 + \frac{p^2}{2} = E$$

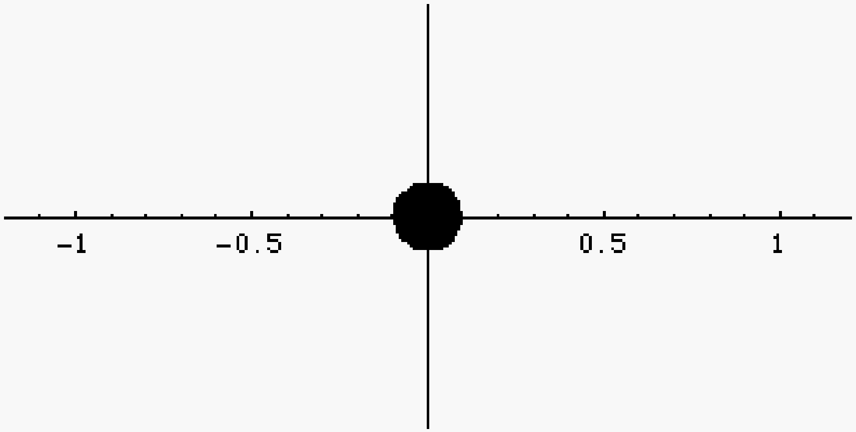
The 'orbit' is an 'attractor'

Phase space of a linear oscillator is a rectangle.

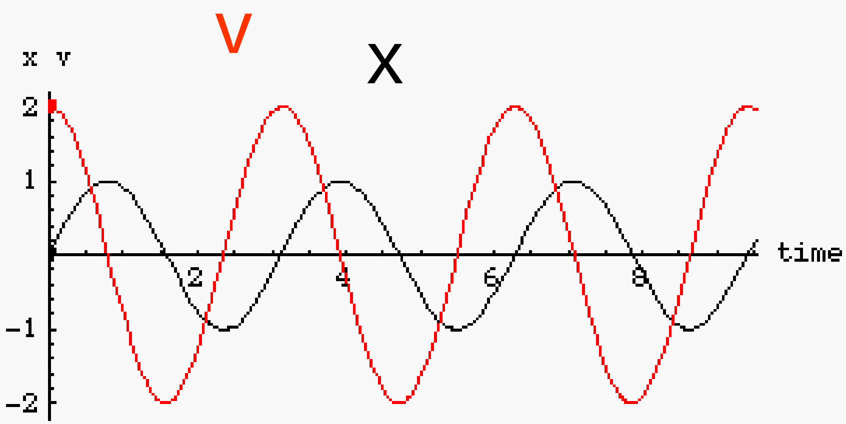


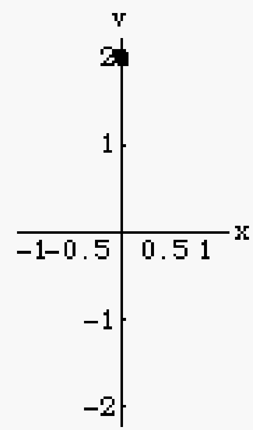
ATTRACTORS 'live' in PHASE SPACE.
An attractor can be a FIXED POINT ("steady state") in phase space, or a periodic orbit ("limit cycles")

Linear Oscillator: ellipse



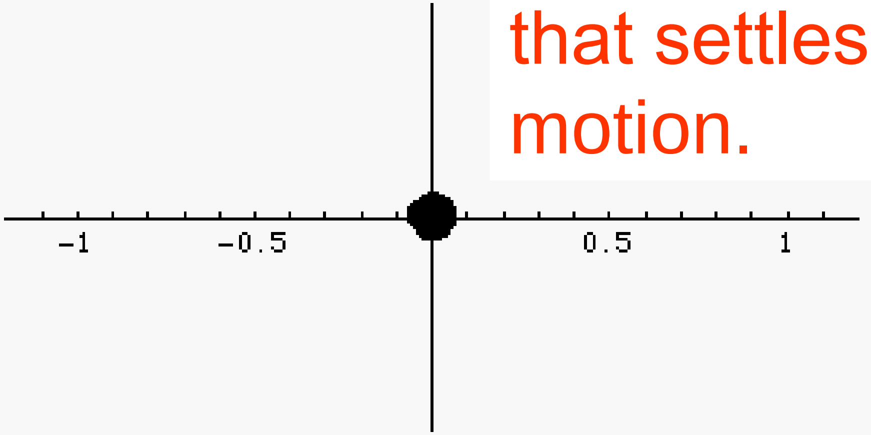
The attractor is a repetitive ORBIT ('limit cycle') in the phase space.



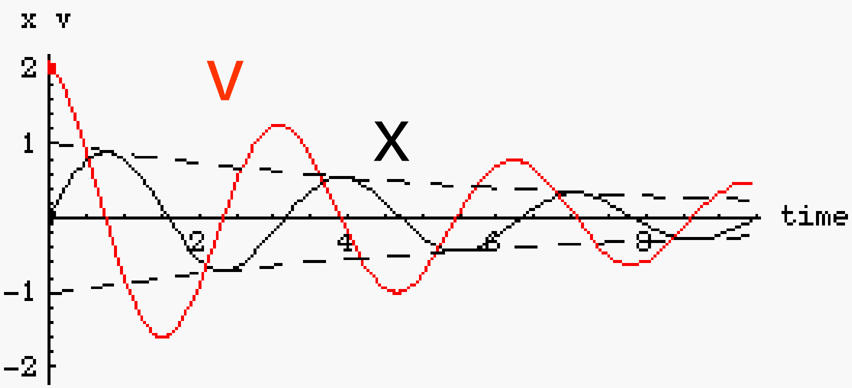


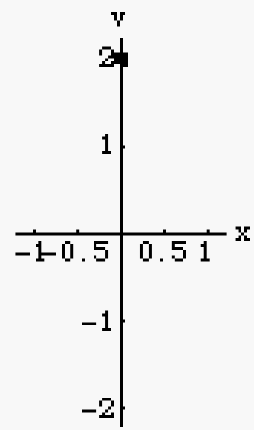
Animation courtesy of Dr. Dan Russell, Kettering University
http://paws.kettering.edu/~drussell/Demos/copyright.html

Damped Oscillator: shrinking ellipse that settles to the ‘steady state’ of no motion.



The attractor is a SINGLE FIXED POINT in the phase space.





Animation courtesy of Dr. Dan Russell, Kettering University
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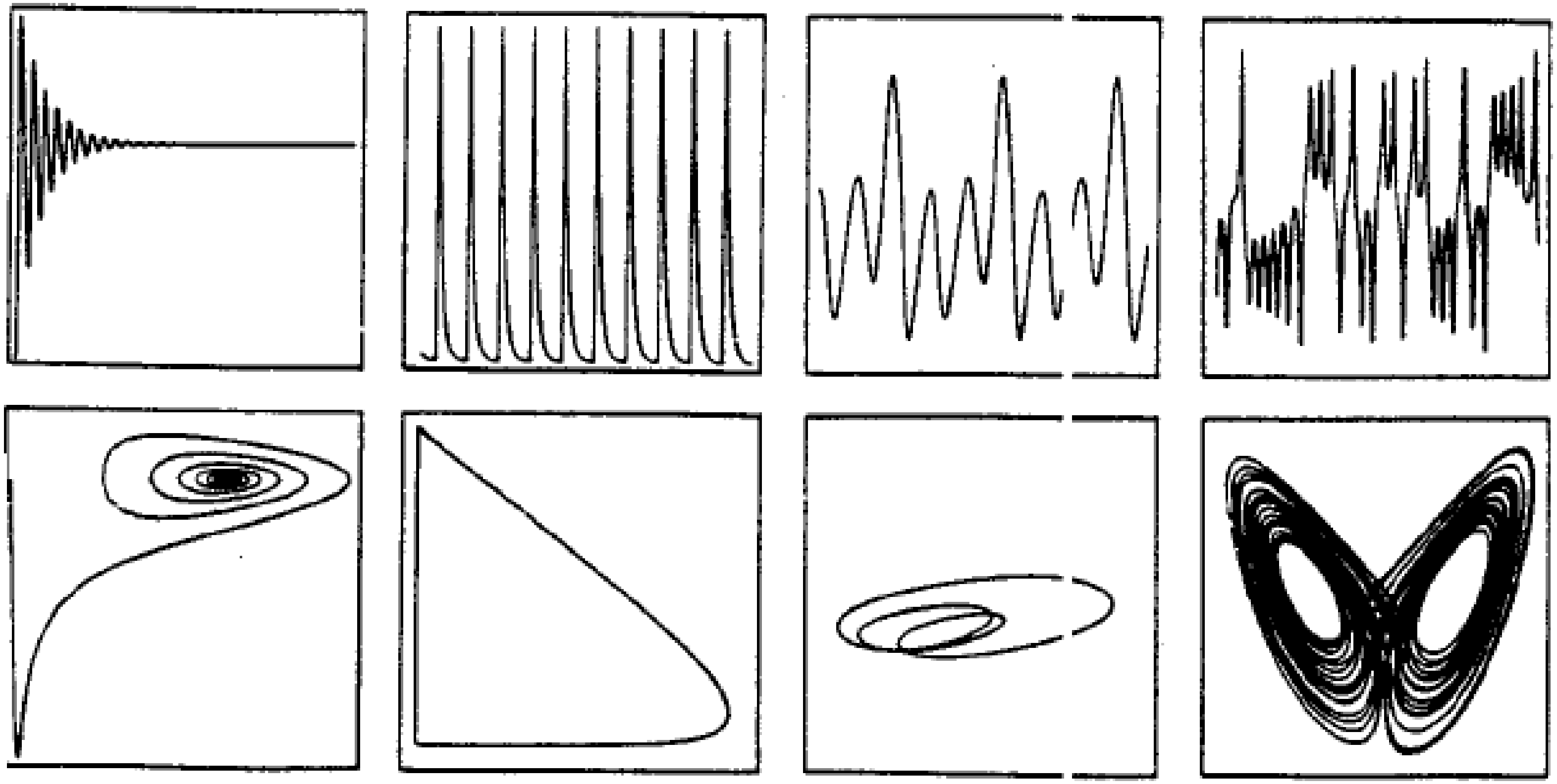
Traditional time series

Phase space trajectory



Steady state

Cyclic orbit

Period three

Chaos

From Gleick's 'Chaos: Making of a new science' page 50

The Lorenz attractor:

$$\frac{dx}{dt} = -\sigma x + \sigma y$$

$$\frac{dy}{dt} = \rho x - y - xz$$

$$\frac{dz}{dt} = xy - \beta z$$

Edward N. Lorenz :

"Deterministic nonperiodic flow"

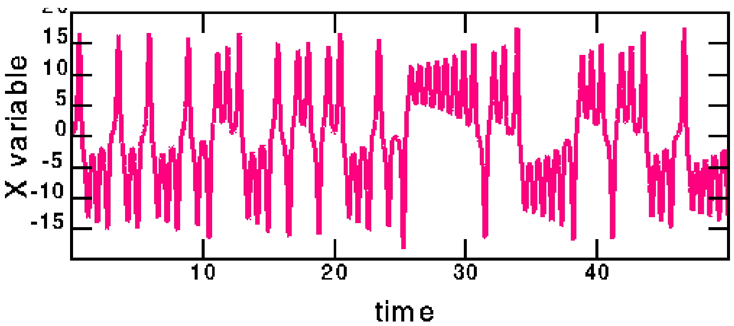
Journal of the Atmospheric Sciences (1963).

A dynamical system described by these equations converges to a ‘strange attractor’ with fractal properties.

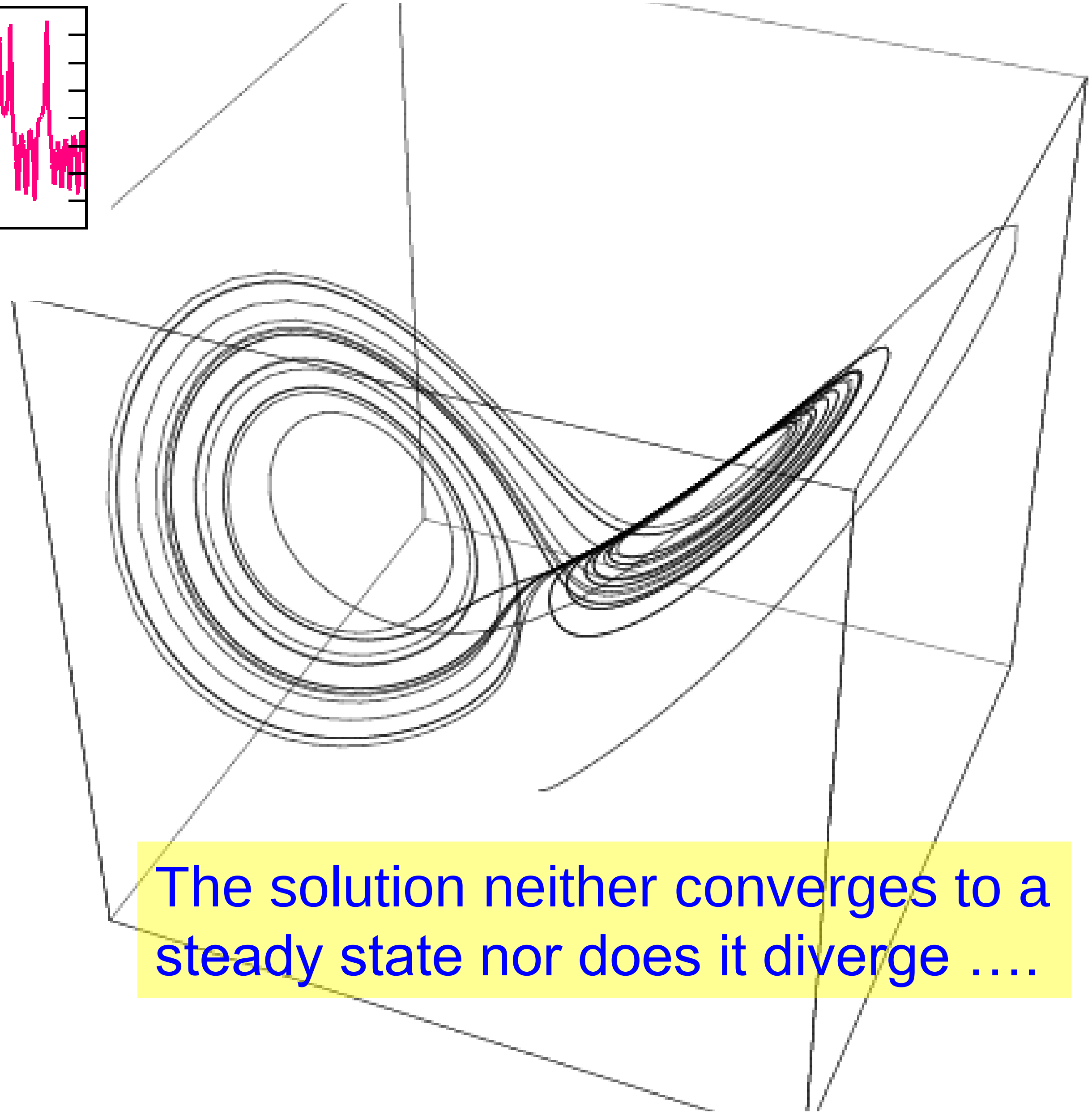
$$example:$$

$$\sigma = 10, \rho = 28, \ \beta = \frac{8}{3}$$

http://www.physics.emory.edu/~weeks/research/tseries1.html#lorenz



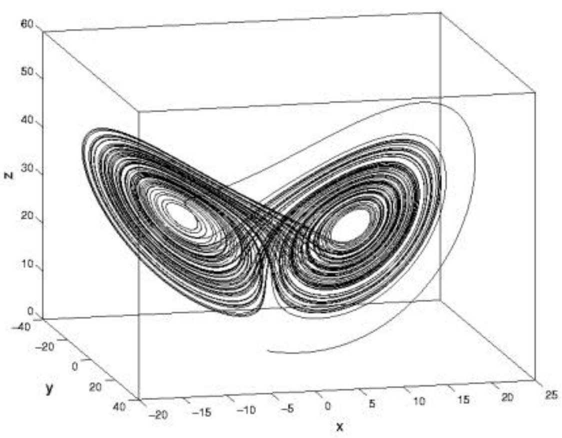
The chaotic dynamical system's motion takes place over a STRANGE ATTRACTOR.



The solution neither converges to a steady state nor does it diverge ….

http://www.tug.org/texshowcase/LorenzAttractor.pdf

The motion of the particle described by a peculiar system of non-linear differential equations such that *the solution will neither converge to a steady state in the phase space, nor diverge to infinity, but will stay in a bounded region*. The trajectory in phase space is nevertheless chaotic, and *sensitive to initial conditions*.

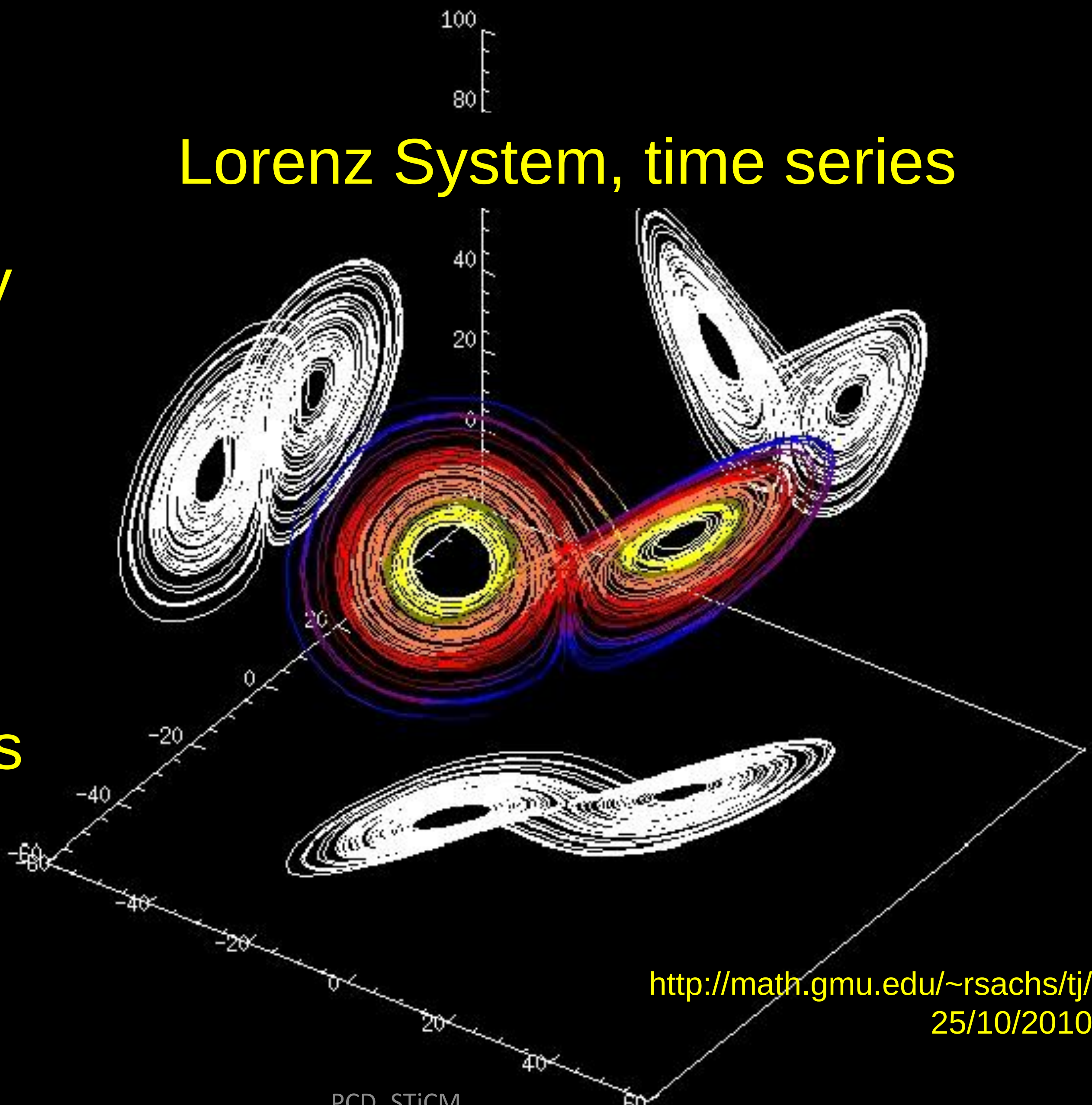


The particle's location, is definitely in the attractor, but is randomly located within the bounded space.

"Order within disorder", since the particle does not leave the "strange attractor".

Lorenz System, time series

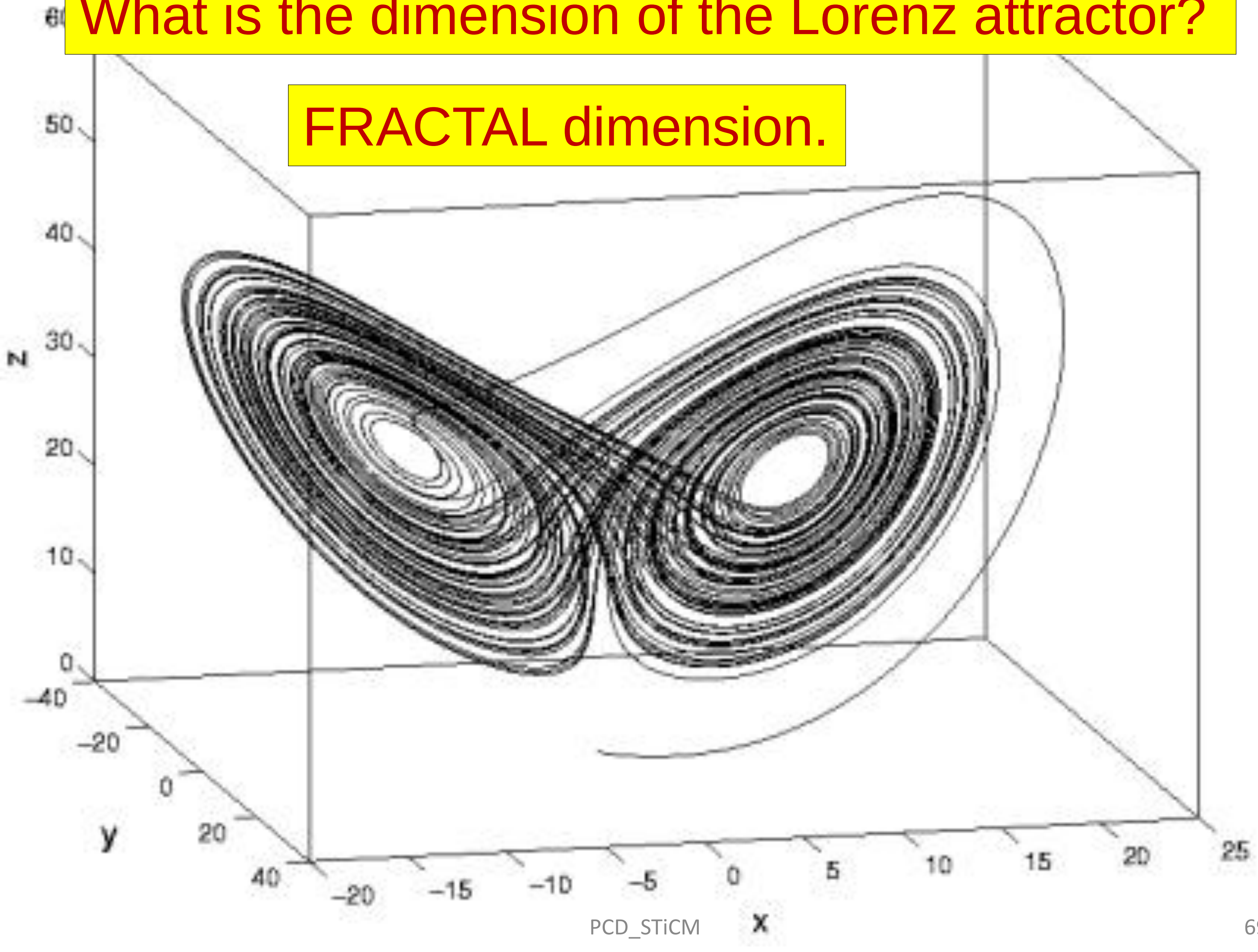
Note the sensitivity of the solution to initial conditions



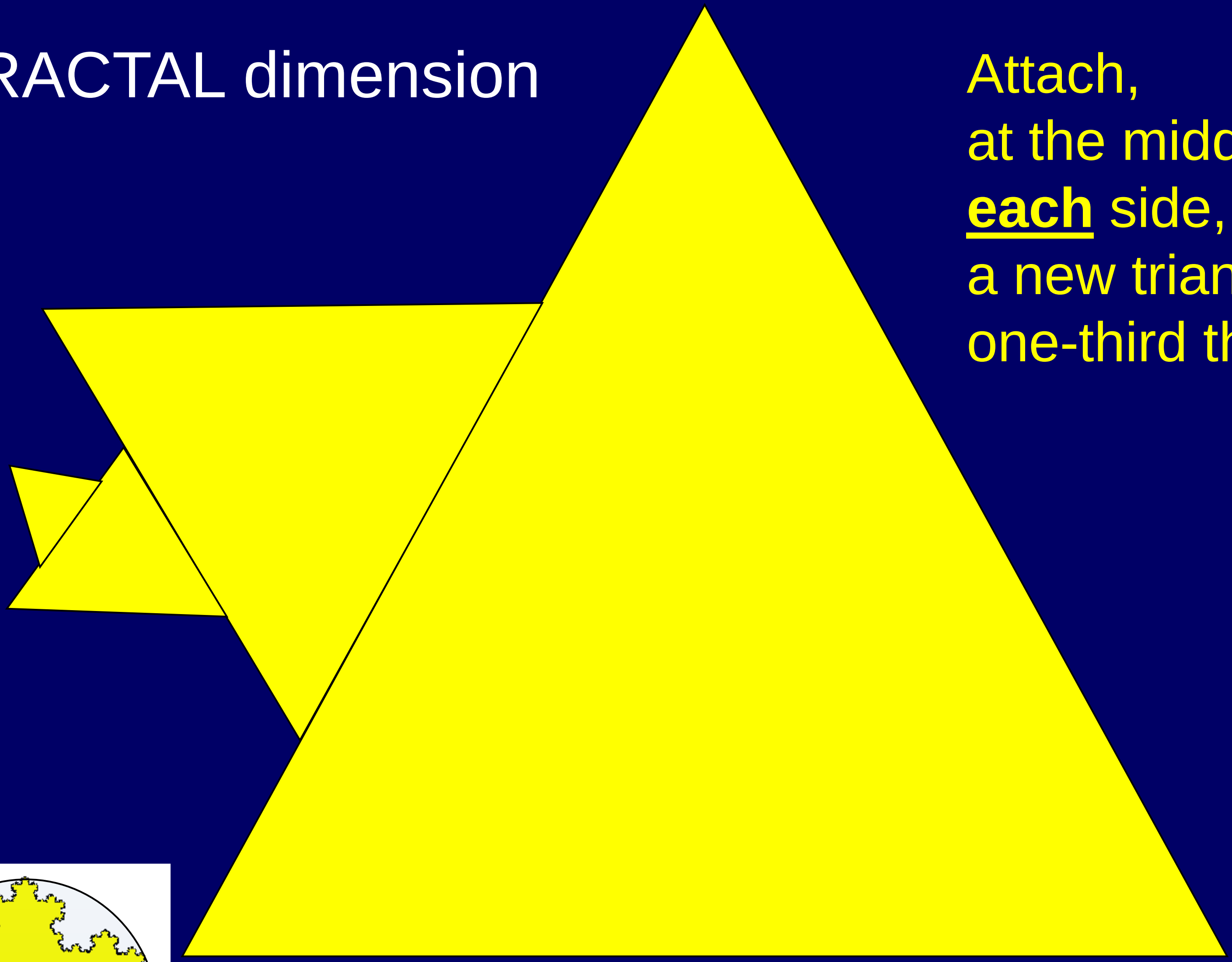
http://math.gmu.edu/~rsachs/tj/
25/10/2010

What is the dimension of the Lorenz attractor?

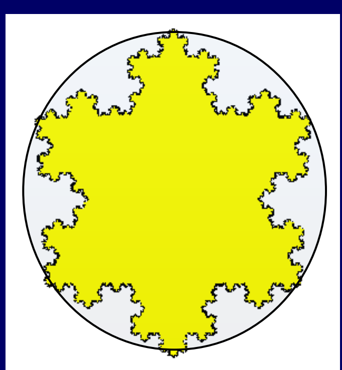
FRACTAL dimension.



FRACTAL dimension



Attach,
at the middle of
<u>each</u> side,
a new triangle
one-third the size



The KOCH snowflakes/ curve

Area < area of the circle drawn around the original triangle

The KOCH snowflakes
or, KOCH CURVE

The perimeter encloses a finite area,

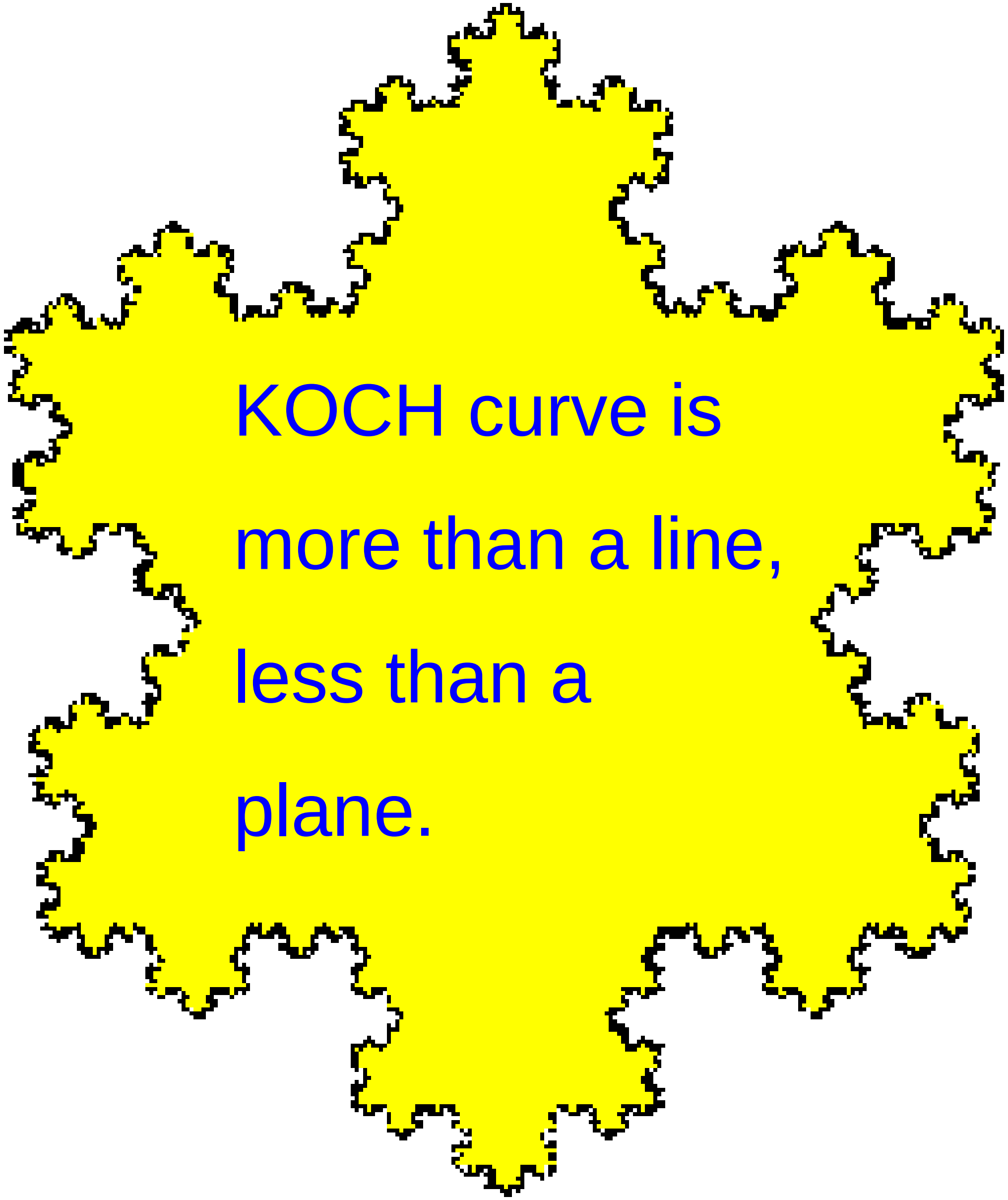
but the length of the perimeter is infinite!

Helge von Koch
Swedish mathematician
described this first in 1904

What is the dimensionality of the Koch curve?

More than 1, less than 2.

Fractal dimension!



STiCM

Select / Special Topics in Classical Mechanics

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STiCM Lecture 38

Unit 11 : **Chaotic Dynamical Systems**

- Fractal Dimensions, Mandelbrot sets

The KOCH snowflakes
or, KOCH CURVE

The perimeter
encloses a finite area,

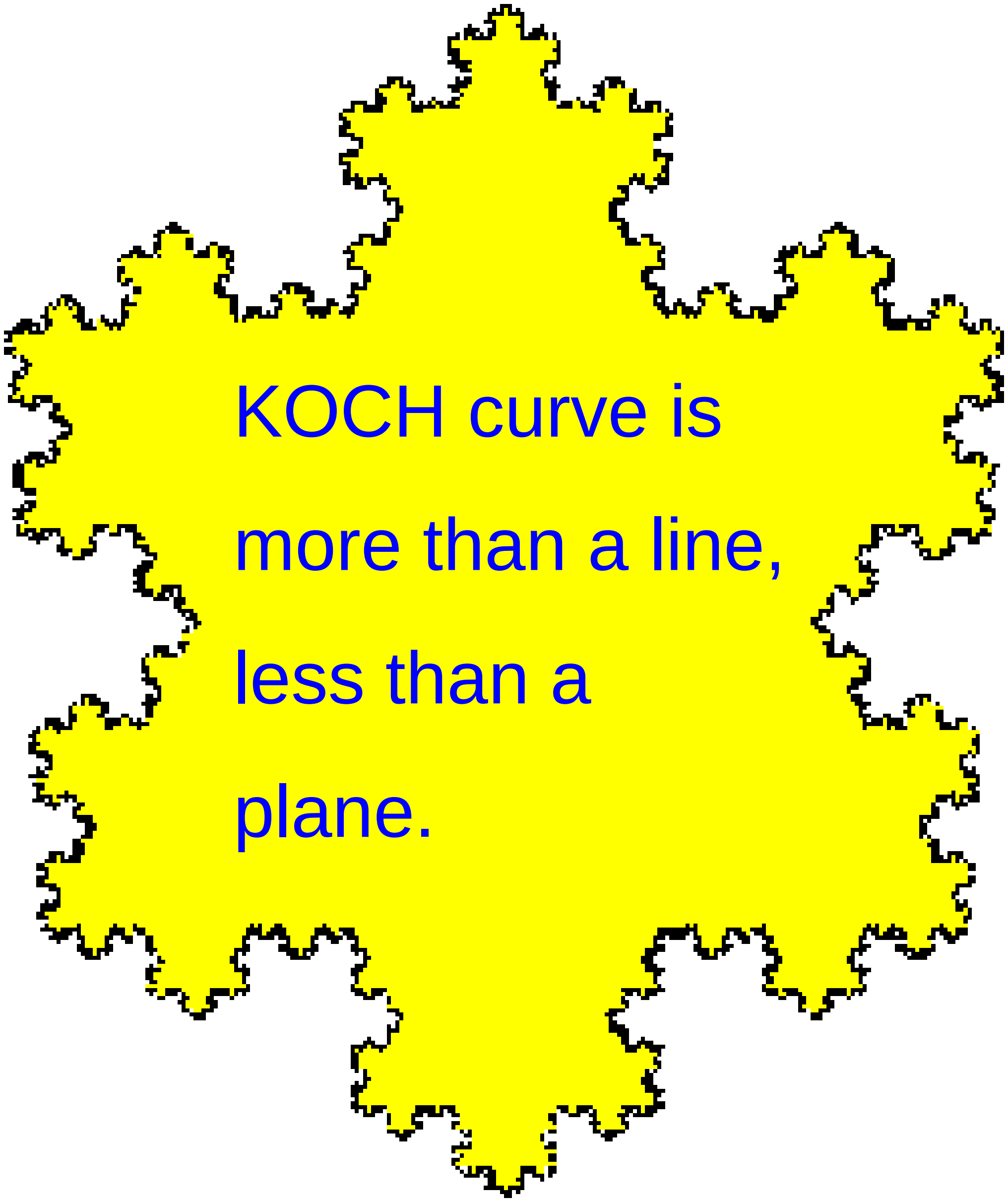
but the length of the
perimeter is infinite!

Helge von Koch
Swedish mathematician
described this first in 1904

What is the dimensionality
of the Koch curve?

More than 1, less than 2.

Fractal dimension!



Felix Hausdorff (1868-1942)

Hausdorff dimension is a mathematical procedure to assign a fractional dimension to a curve or shape.

Hausdorff-Besicovitch dimension.

Fractal: is a set for which the Hausdorff-Besicovitch dimension exceeds the topological dimension.

Topological dimension:

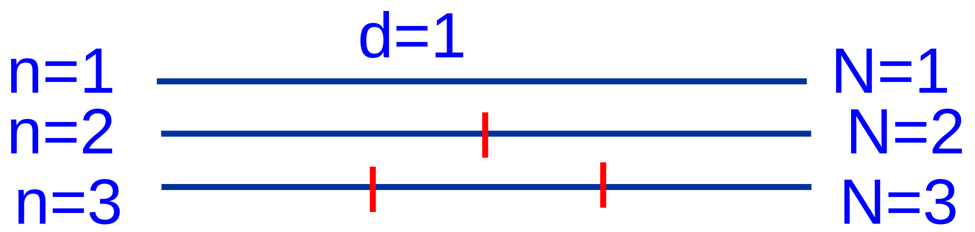
point : 0-dimensional; line : 1-dimensional;

a plane : 2-dimensional; Euclidean space R^n :*n*-dimensional.

Dimension of space = no. of real parameters needed to describe different points in that space.

This idea breaks down!

Cantor's work (also Peano's): There is a one-to-one correspondence between R^1 and R^2.

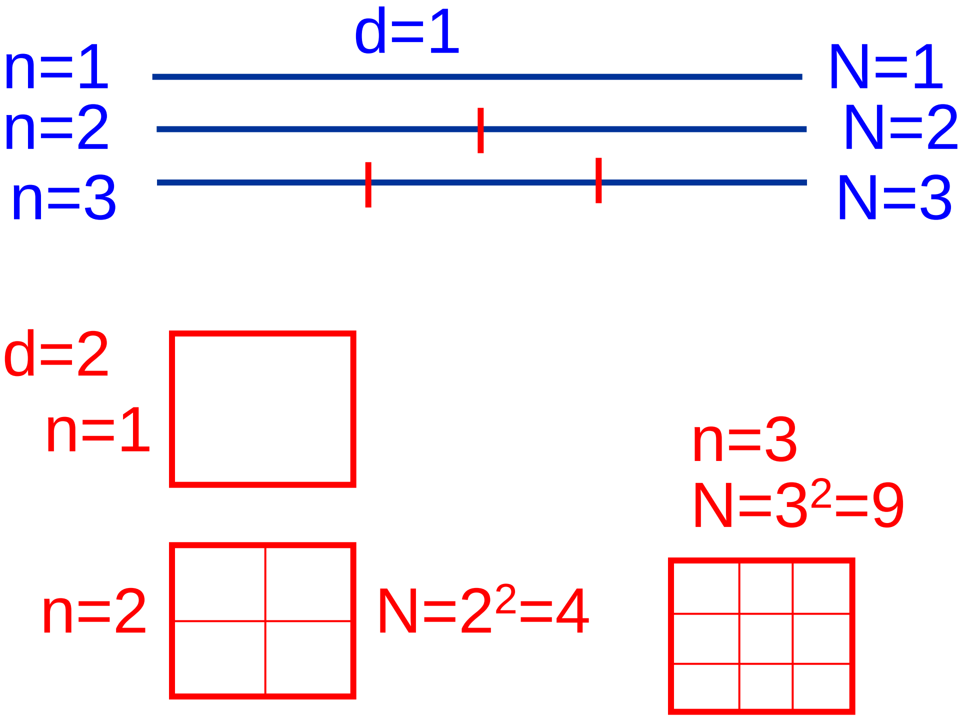


Take an object in Euclidean one dimension.

Reduce this dimension by a factor of n.
Cut it in n pieces.

The number of individual units we then have is $N=n^d$,

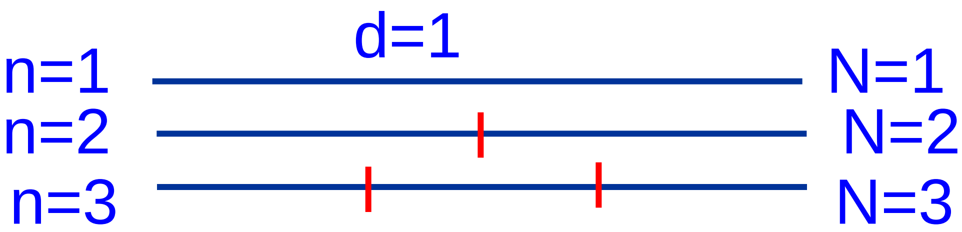
In this case, d=1 is the dimension.



Take an object in Euclidean dimension d.

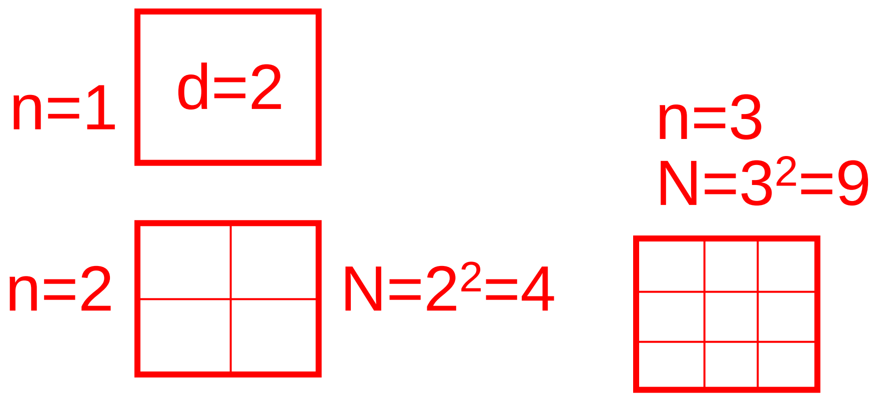
Reduce ***each*** dimension by a factor of ***n***.
i.e.,
cut each side into n pieces.

The number of individual units we then have is $N=n^d$.



Take an object in Euclidean dimension d.

Reduce each dimension by a factor of n. Cut it in n pieces.



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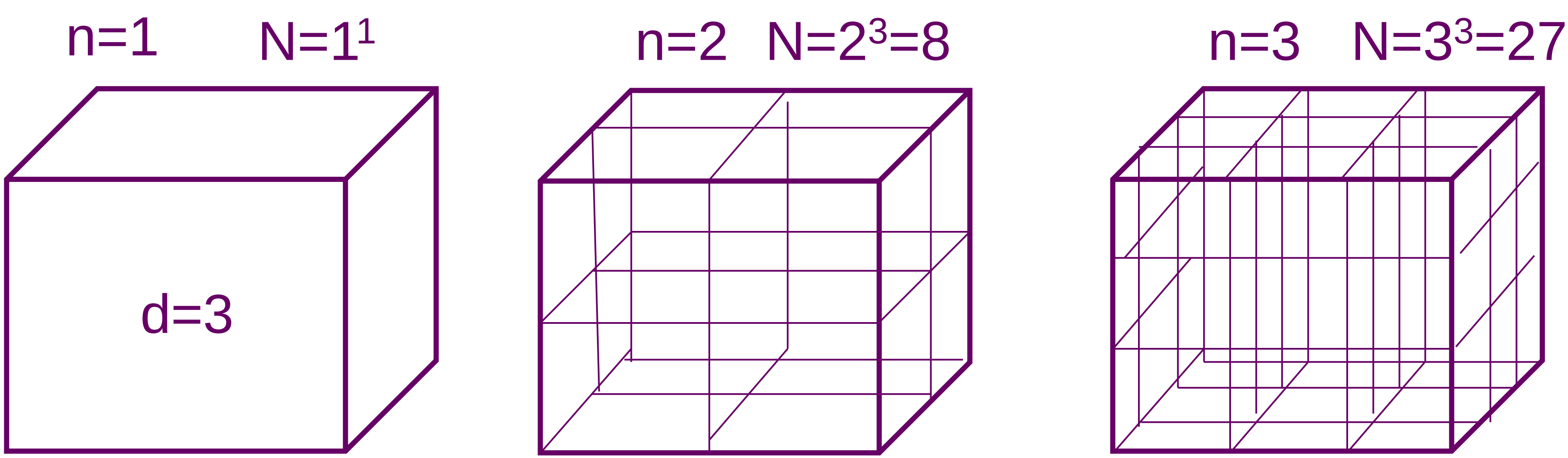


Take an object in Euclidean dimension d.

Reduce each dimension by a factor of n. Cut it in n pieces.

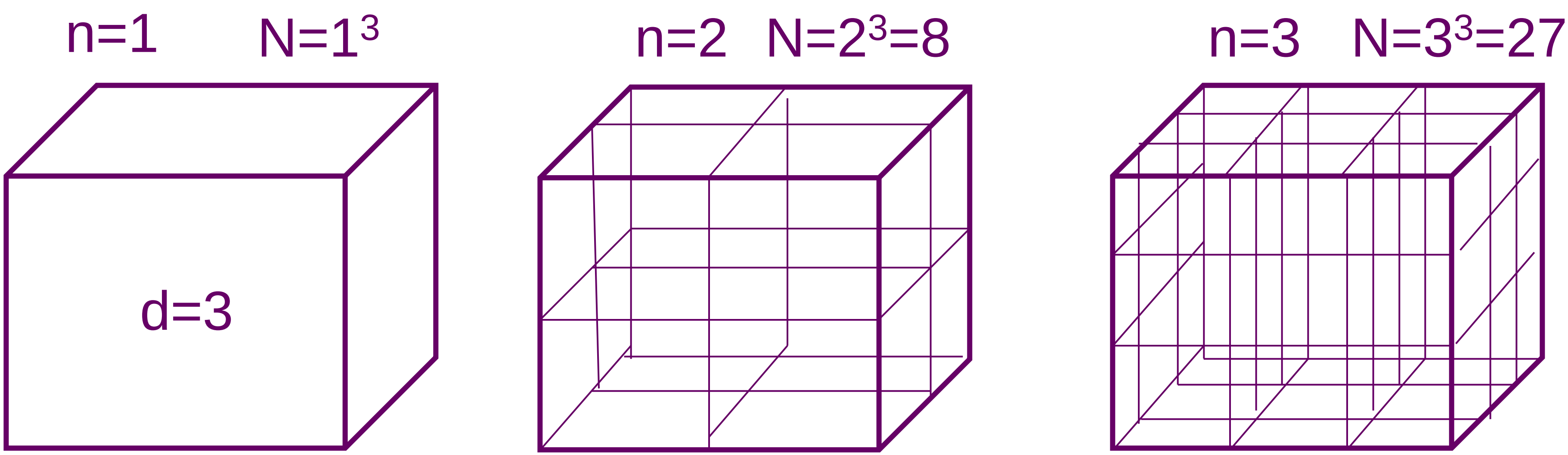
The number of individual units we then have is $N = n^d$.

$$\log N = d \log n$$

$$d = \frac{\log N}{\log n}$$

dimensionaility d

need *NOT* be an integer, it can be a fractional number



Benoît Mandelbrot
Born: 20th Nov. 1924
Polish; moved to France
French-American

Father of
FRACTAL GEOMETRY

What is the length of the coast line of Great Britain?
How would you measure it?

Measurement of length: Lay down lots of straight-line rulers/scales and count the number of scales, add them up

scales

If we use a scale of half the previous length, we need more than twice the number of scales.

Each successive time we use smaller scale to get more accurate answer, we get a longer length.

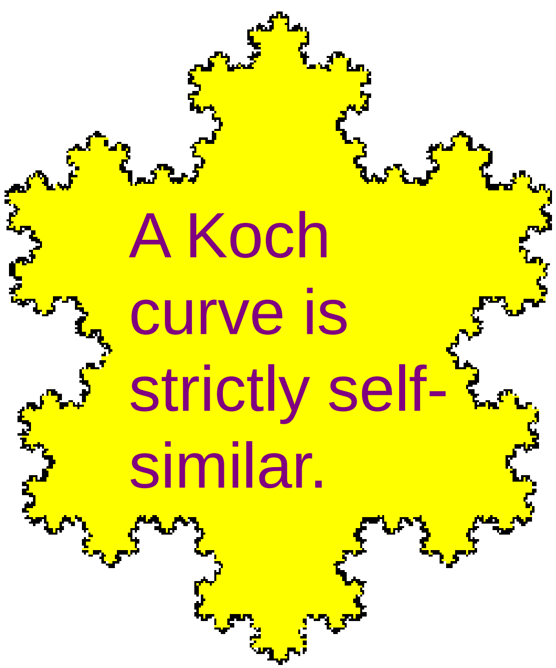
What is the length of the coast line of Great Britain?

How would you measure it?

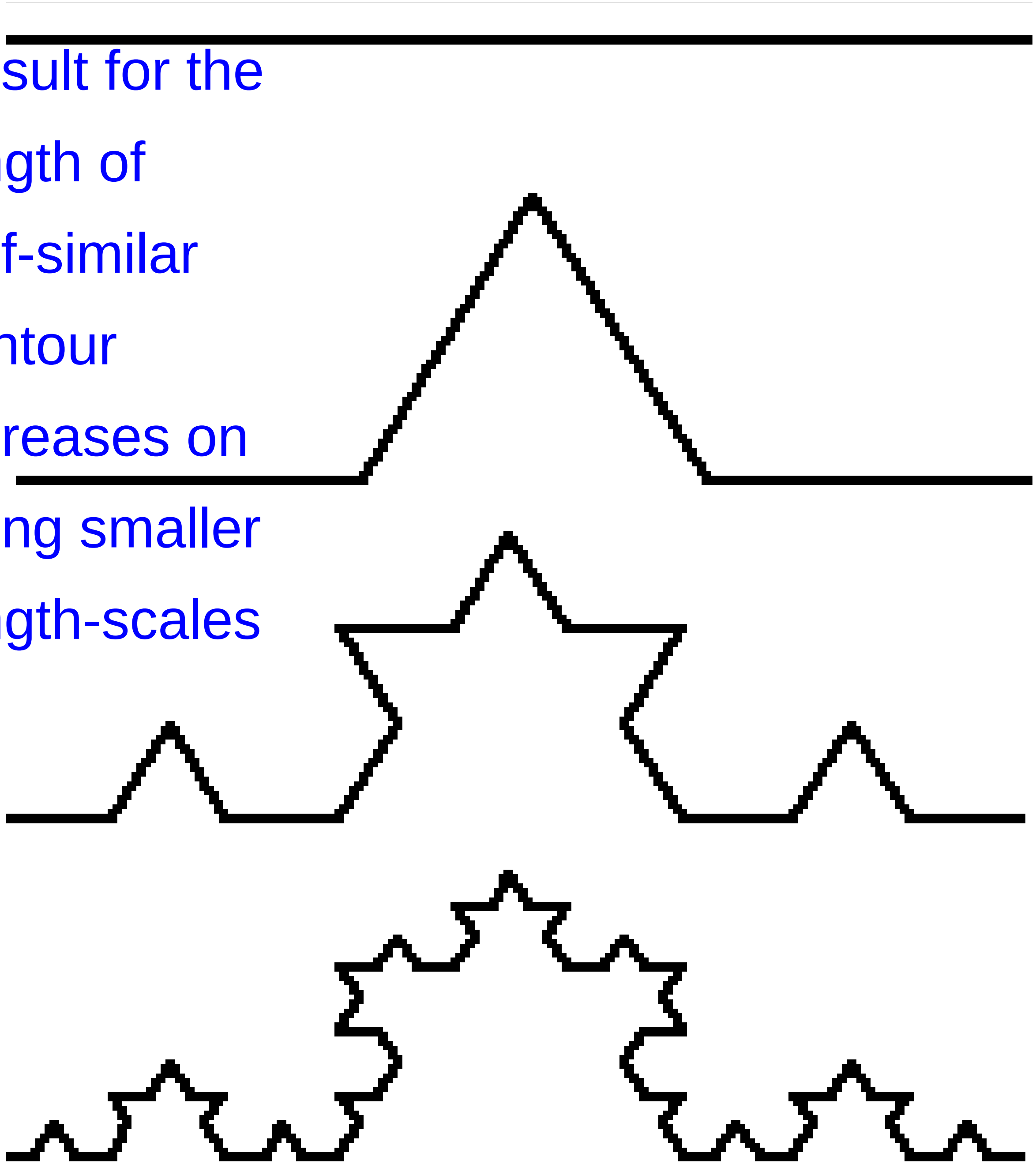
With smaller scales, one can reach various nooks and corners

The Coastline has only a hint of ‘self-similarity’.

Big bays and peninsulas contain mid-sized bays and peninsulas in them, and these have in turn many small bays and peninsulas.



Result for the length of self-similar contour increases on using smaller length-scales



1 segment of unit length
Length=1

4 segments of one-third unit:
Length=4/3

16 segments of one-ninth unit:
Length=16/9

64 segments of one-twenty-seventh unit:
Length=64/27

Iteration Number	Segment Length	Number of segments	Curve Length
1	1	1	1.00
2	1/3	4	1.33
3	1/9	16	1.77
4	1/27	64	2.37

Each side is broken into 4 smaller pieces, with a magnification factor of 3.

1

2

3

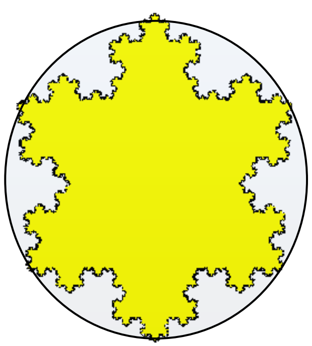
4

Dimensionality of the KOCH curve

$$d = \frac{\log N}{\log n} = \frac{\log 4}{\log 3} = 1.261....$$

"Fractal dimension" is sometimes called "Similarity dimension"

"Fractal dimension" is defined for those sets that are affine "self-similar"



The KOCH snowflakes/curve

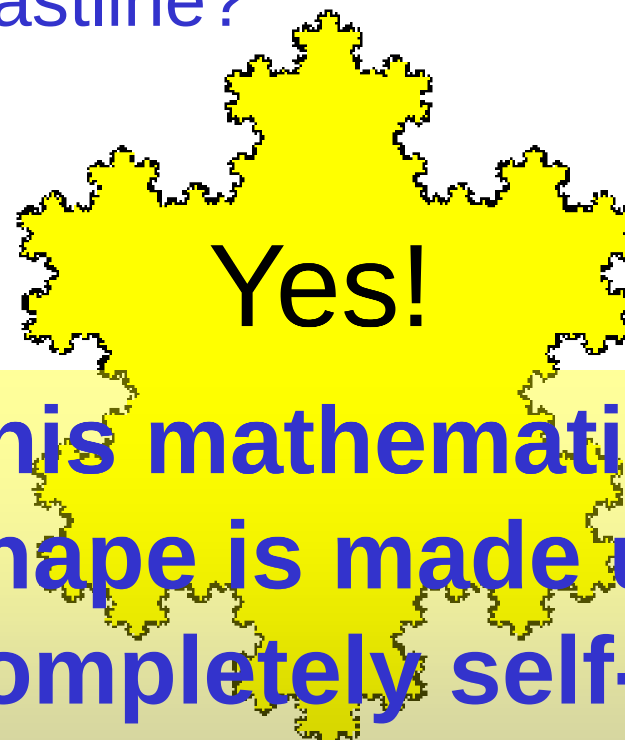
Each successive time we use smaller scale to get more accurate answer, we get a longer length for the coastline.

Will successive measurements with smaller scales give an infinite length for the coastline?

No!

'self-similarity' breaks down at some level.

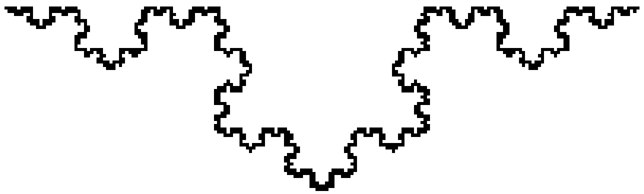
Coastline is made of finite discrete matter.

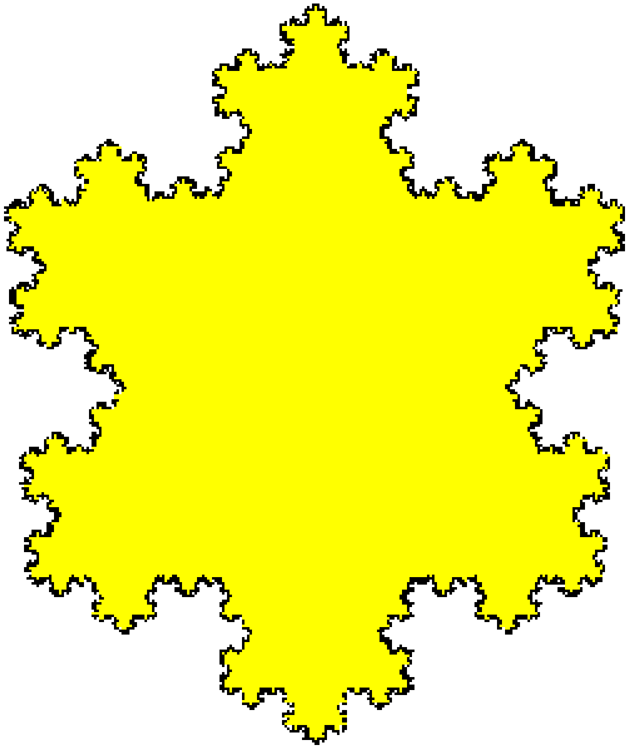


Yes!

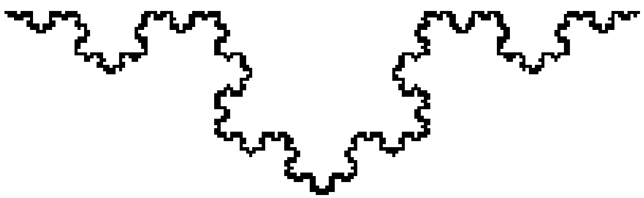
This mathematical shape is made up completely self-similar segments.

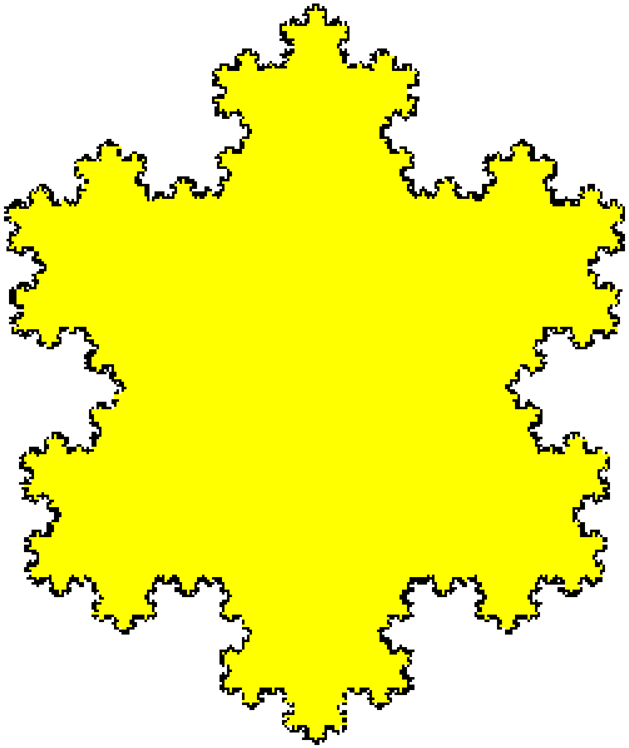
Infinite SELF-SIMILARITY of the KOCH CURVE





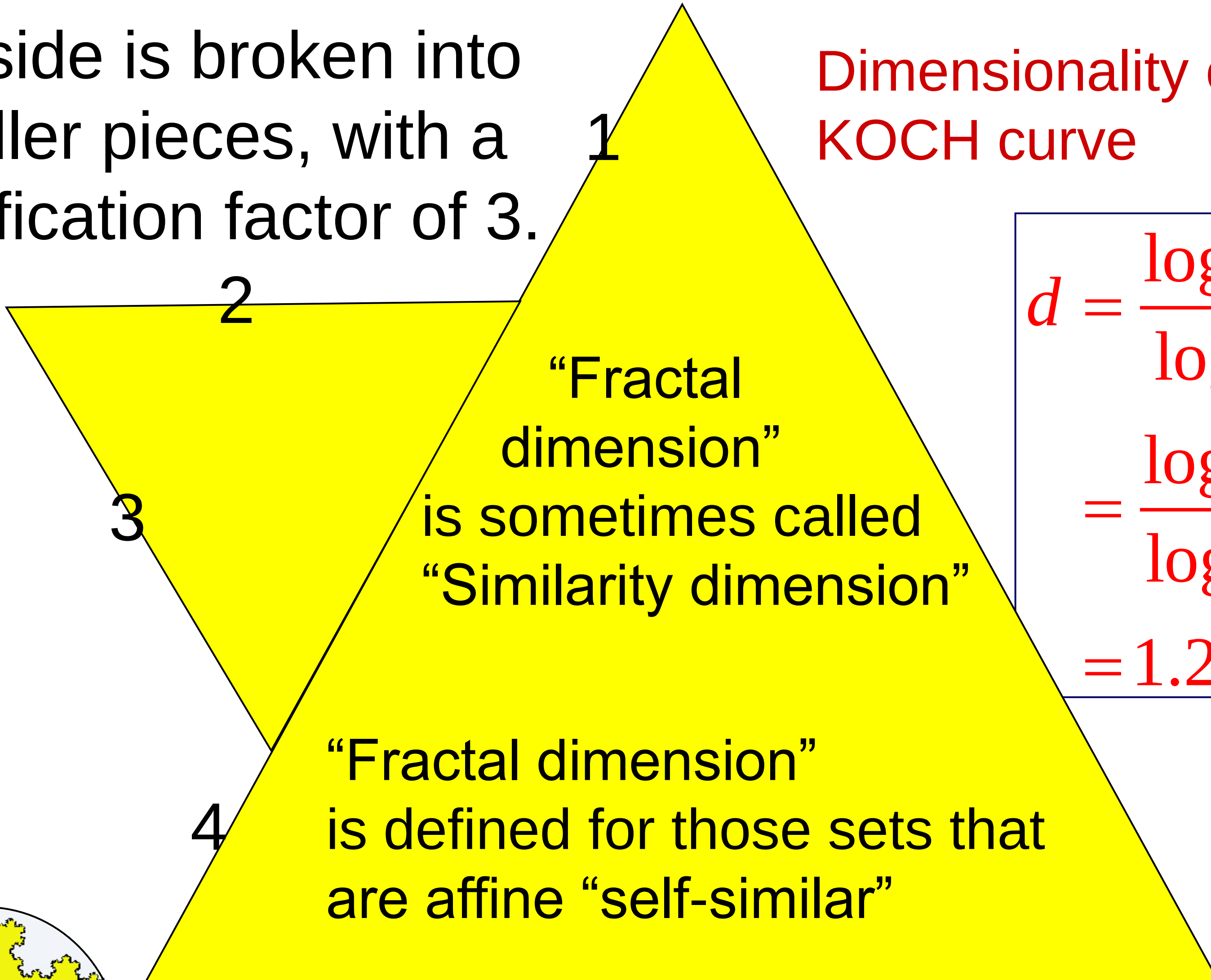
Infinite SELF-SIMILARITY of the KOCH CURVE





Each side is broken into 4 smaller pieces, with a magnification factor of 3.

Dimensionality of the KOCH curve



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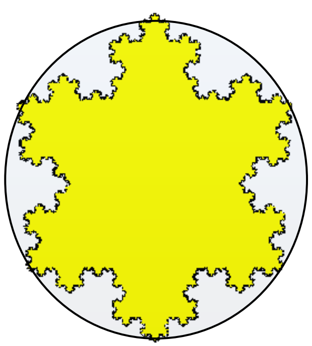


The KOCH snowflakes/curve

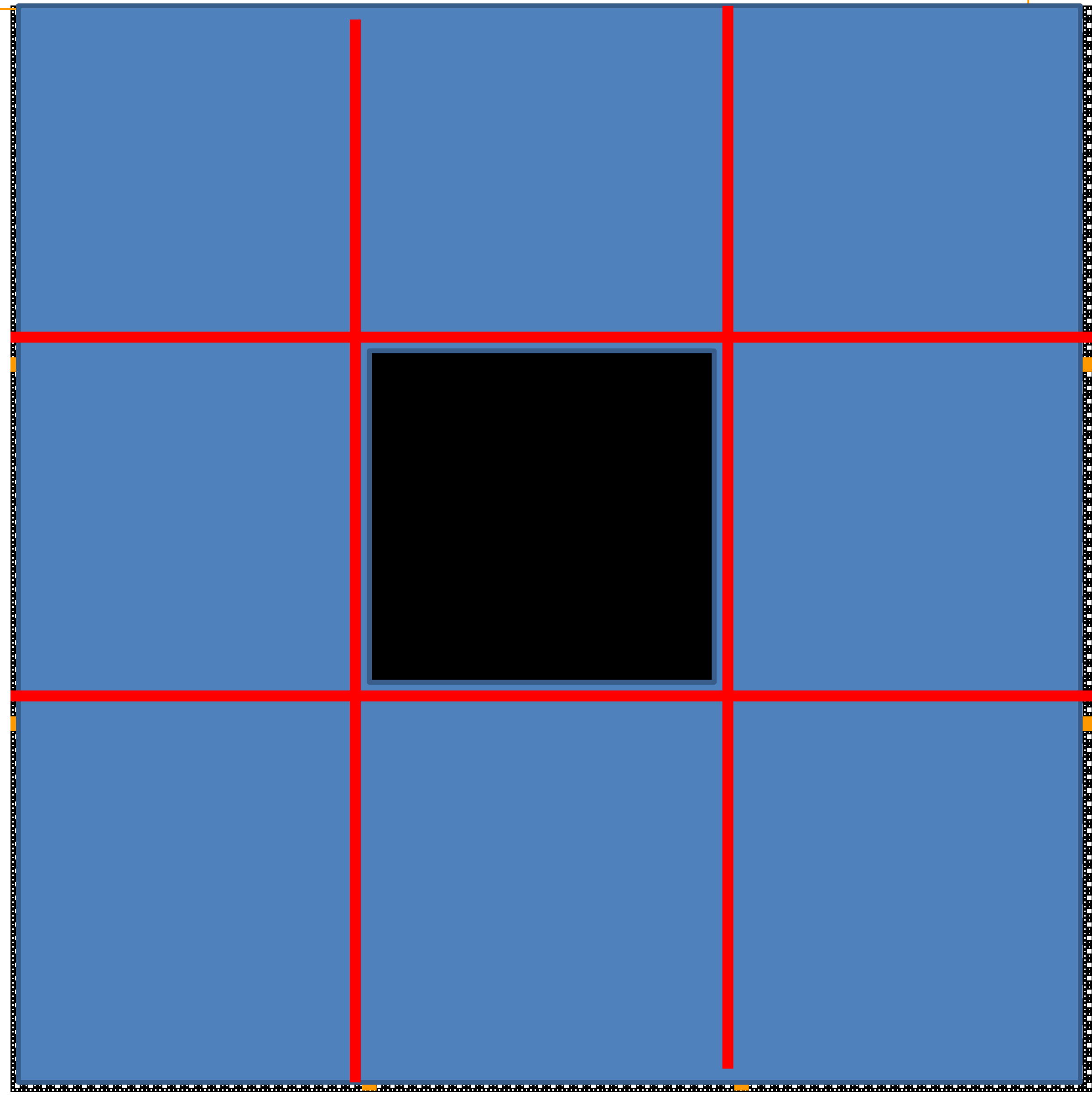
(Waclaw) Sierpinski carpet (1916): a plane fractal

Begin with a square.

Divide it into 3x3=9 equal squares.

Remove the central square.

Repeat (self-similar) successively on remaining squares by putting ‘square-holes’ in the center.



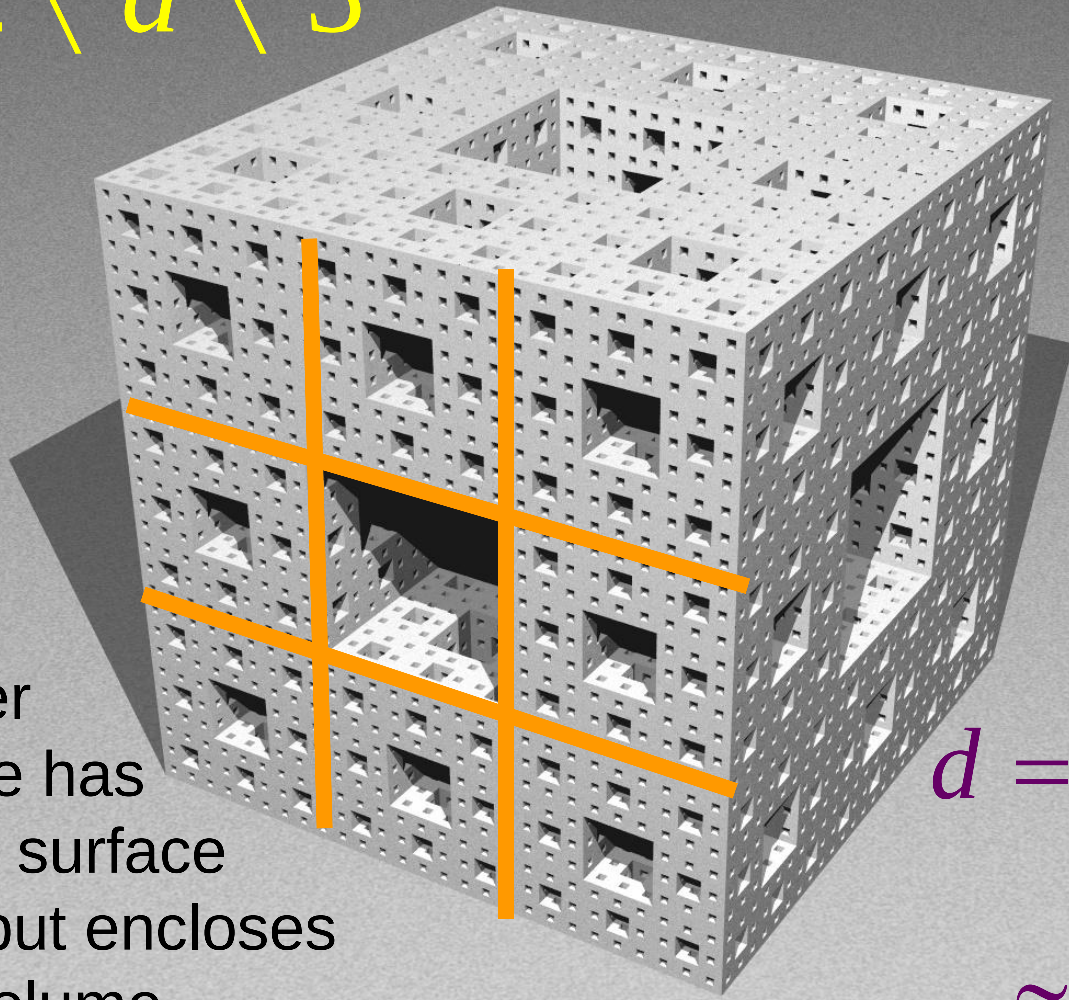
Dimensionality of Sierpinski carpet:

$1 \langle d \langle 2$

Hausdorff ‘self-similar’ ‘fractal’ dimension

Menger sponge: 3-dimensional analogue of Sierpinski carpet

$$2 \langle d \langle 3$$



Menger sponge has infinite surface area, but encloses zero volume.

$$d = \frac{\log 20}{\log 3} \approx 2.7268$$

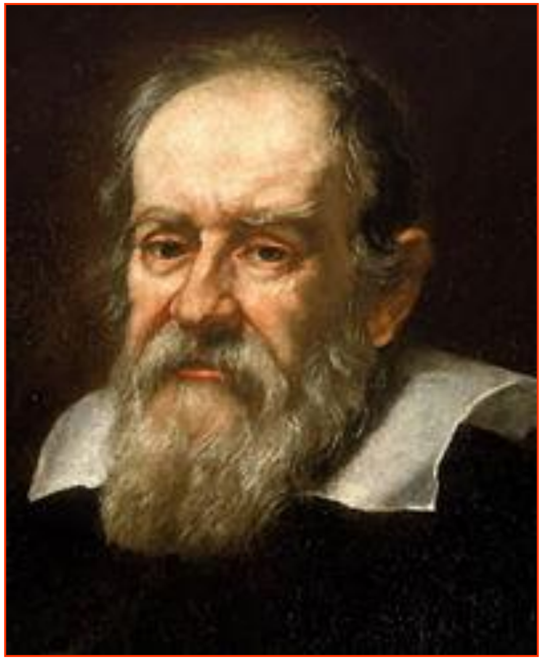
Nature is discrete. Mathematics is not constrained by nature.

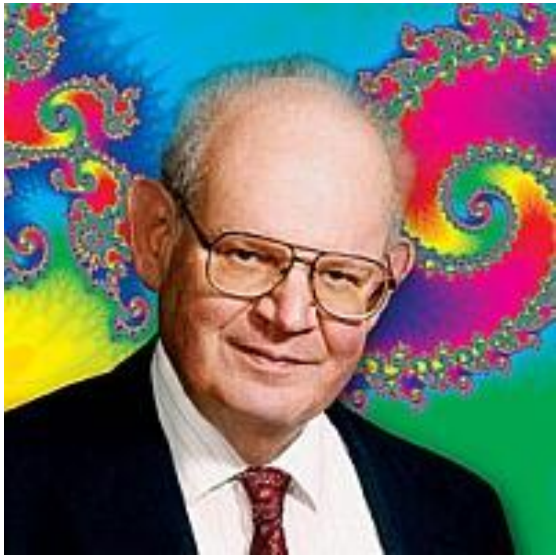
One can have a mathematical shape that has an infinite perimeter but a finite area, or infinite area that would enclose only a finite volume.

You will get a finite perimeter length if you use a rigid ruler to measure the perimeter. A smaller ruler will yield a bigger value for the length of the perimeter. This growth continues without converging to any finite value as you keep making the ruler smaller.

"…. the universe …. Cannot be understood unless one first learns to comprehend the language in which it is written. It is written in the language of mathematics, and its characters are triangles, circles and other geomteric figures,…."

– Galileo Galilei (in 1623)





"….Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth,…."

– Benoit Mandelbrot (in 1984)

"Iteration" / "Orbit"

Iterations

$$x_0 : \text{ seed value}$$

$$x_1 = F^1(x_0) = F(x_0)$$

$$x_2 = F^2(x_0) = F(F(x_0))$$

"To Iterate" = to evaluate the function over and over again, using the output of the previous step as input for the next.

Orbit of $x^2 - 2$ *for different seed values* x_0

$n=0$	$x_0=0$	$x_0=0.1$
$n=1$	-2	-1.99
$n=2$	2	$+1.960$
$n=3$	2	1.842
$n=4$	2	1.393
$n=5$	2	-0.597

Orbit for seed 0 gets eventually 'fixed', but for neighboring seed point 0.1, the orbit wanders between -2 and +2 randomly.

A fixed point *orbit is one for which* $F(x_0) = x_0$.

STiCM

Select / Special Topics in Classical Mechanics

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STiCM Lecture 39

Unit 11 : **Chaotic Dynamical Systems**

- Bifurcation, Chaos, Mandelbrot sets

Each side is broken into 4 smaller pieces, with a magnification factor of 3.

1

2

3

4

Dimensionality of the KOCH curve

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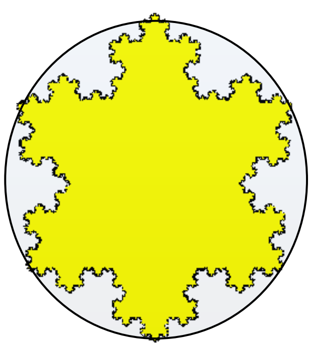
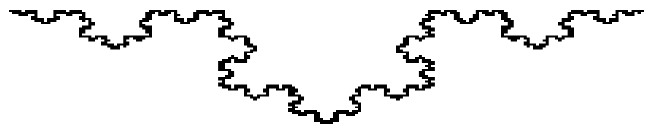
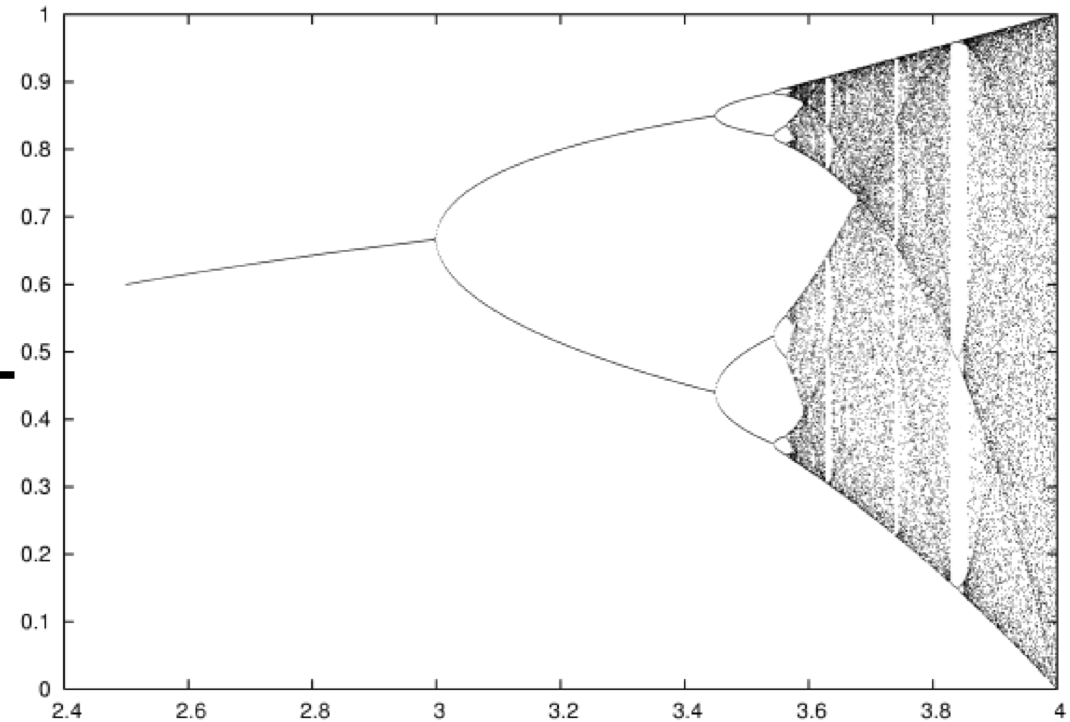
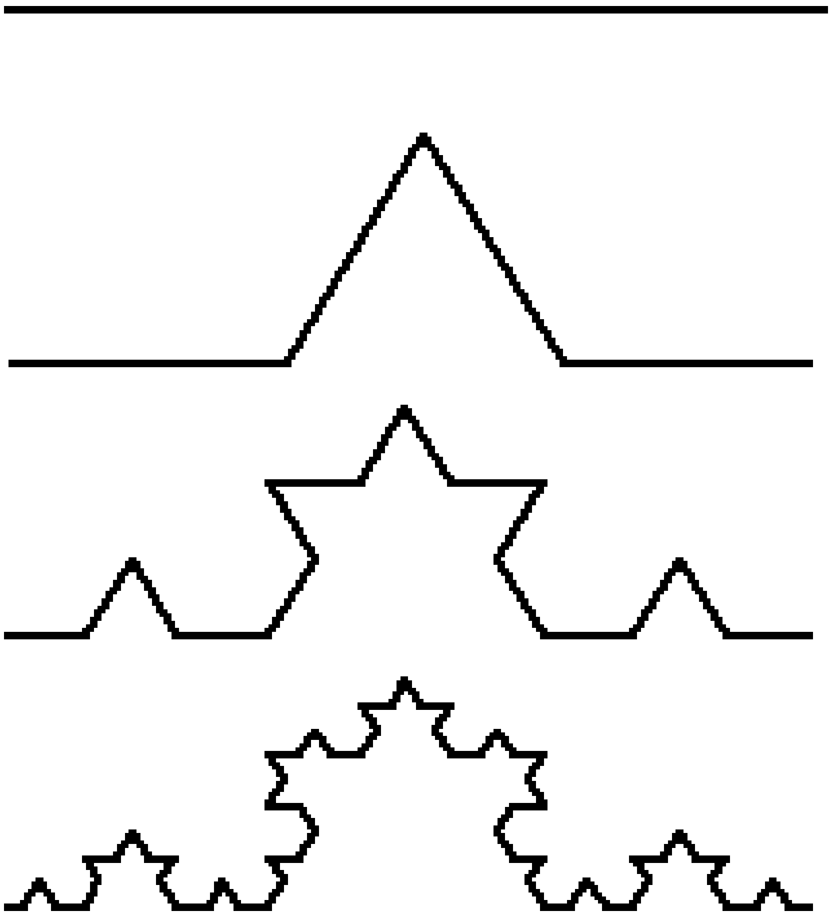


The KOCH snowflakes/curve



SELF-SIMILARITY: important aspect of 'CHAOS'





Feigenbaum discovered the exact scaling factor (4.669…..) at which it was self-similar.

Iterations

$$x_0 : \ seed \ value$$

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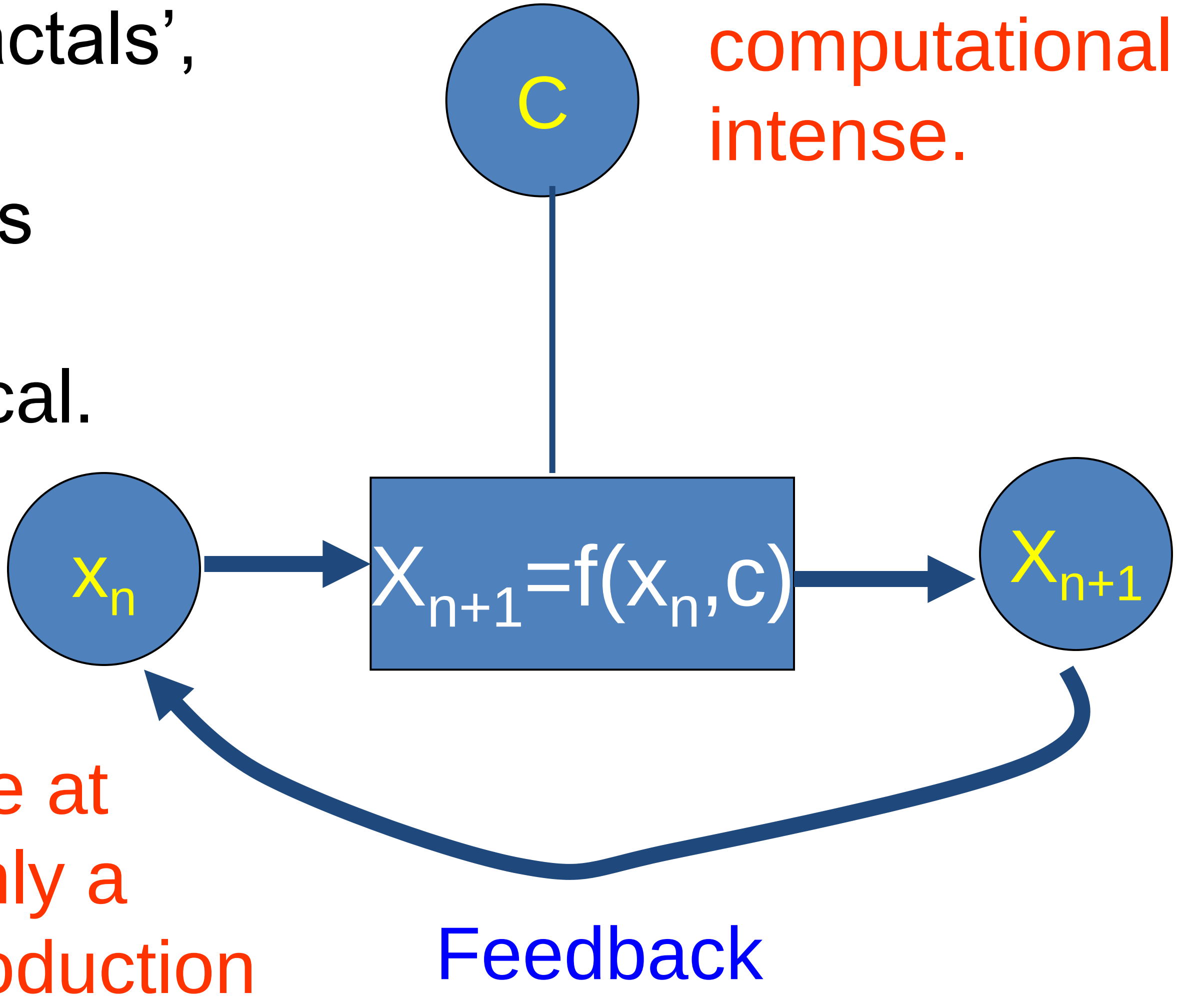
A fixed point *orbit is one for which* $F(x_0) = x_0$.

The subject of ‘chaos’, ‘fractals’, ‘non-linear dynamics’ is intensely mathematical.

Also, it is computationally intense.



We aim here at providing only a cursory introduction without using heavy numerical/ computational, mathematical techniques.

Mandelbrot set: set of all complex numbers z for which sequence defined by the iteration

$z(0) = c,\ z(n+1) = z(n)*z(n) + c,\qquad n=0,1,2,3,\ldots$

remains bounded.

If $c=0$, then $z(n) = 0$ for all n, so the limit of the sequence is zero.

If $z=i$, the sequence oscillates between i and $i-1$, so the sequence remains bounded without converging to a limit.

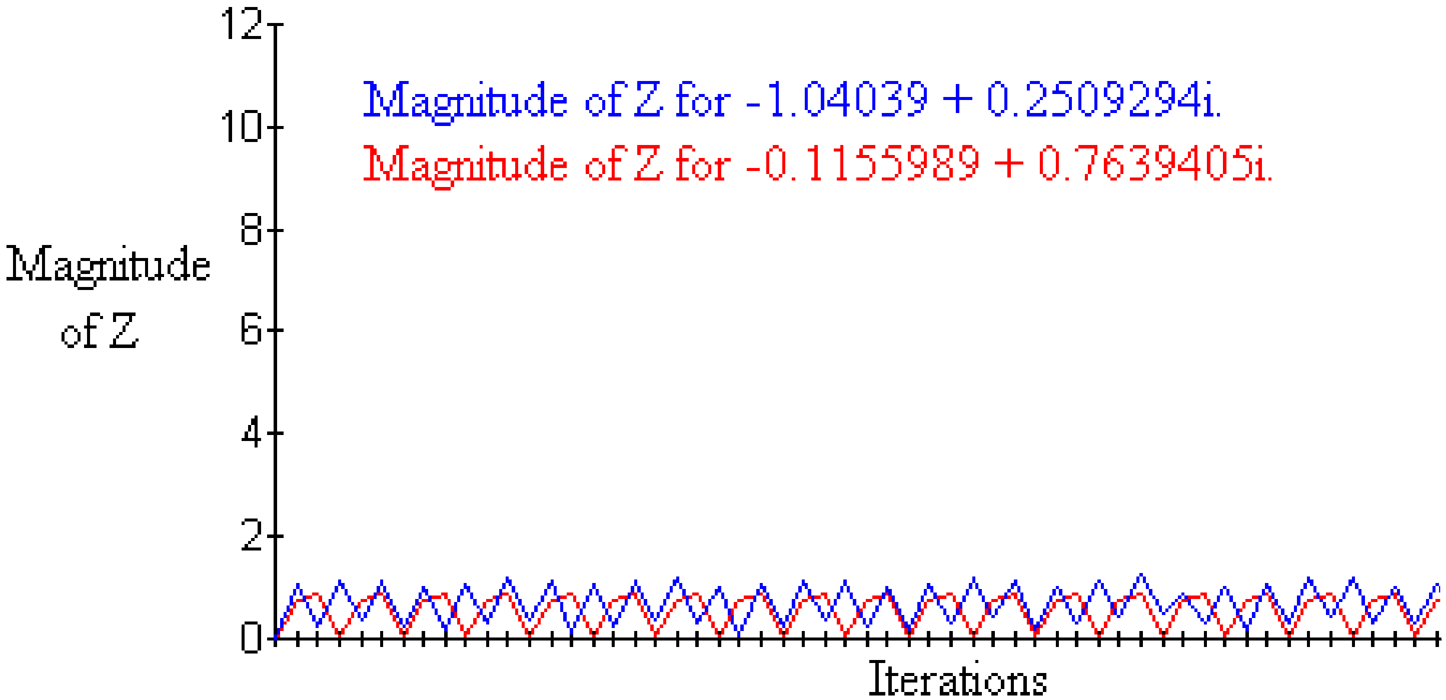
Mandelbrot set:

$$z = z^2 + c$$
$$c : complex\ number$$

Do the same to the complex number that results from the above operation.

i.e. ITERATE: *If the functions* ***g(z) = z² + c*** *are used to do the iterations, then which values of* ***c*** *give orbits that escape, and* ***which values of c give orbits that do not escape****?*

If the result tends to infinity, exclude c; if the result of a large number of iterations stays below a certain level, include ‘c’ as part of ‘Mandelbrot set’.



Introduction to the Mandelbrot Set

A guide for people with little math experience.

By David Dewey

http://www.ddewey.net/mandelbrot/

2,60,463

If |Z| > 2, it will escape to infinity.

That is, we don't have to check it for infinity, just for 2.

How many times should we iterate Zn to see if it goes farther away than 2 or not?

Luckily just a few times suffices.

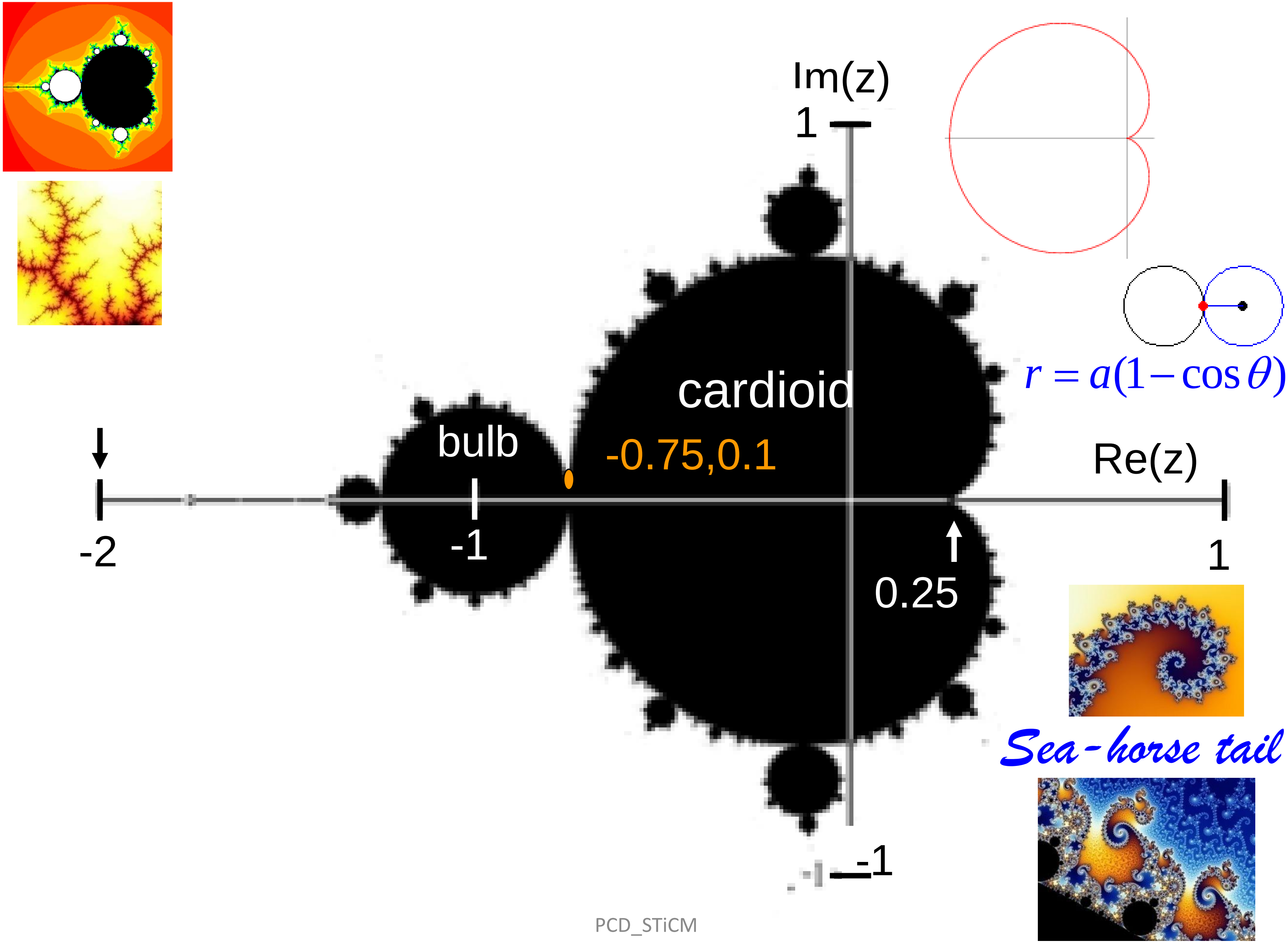
The Mandelbrot set is a ***fractal***.

Fractals: objects that display self-similarity at various scales.

Magnifying a fractal reveals small-scale details similar to the large-scale characteristics.

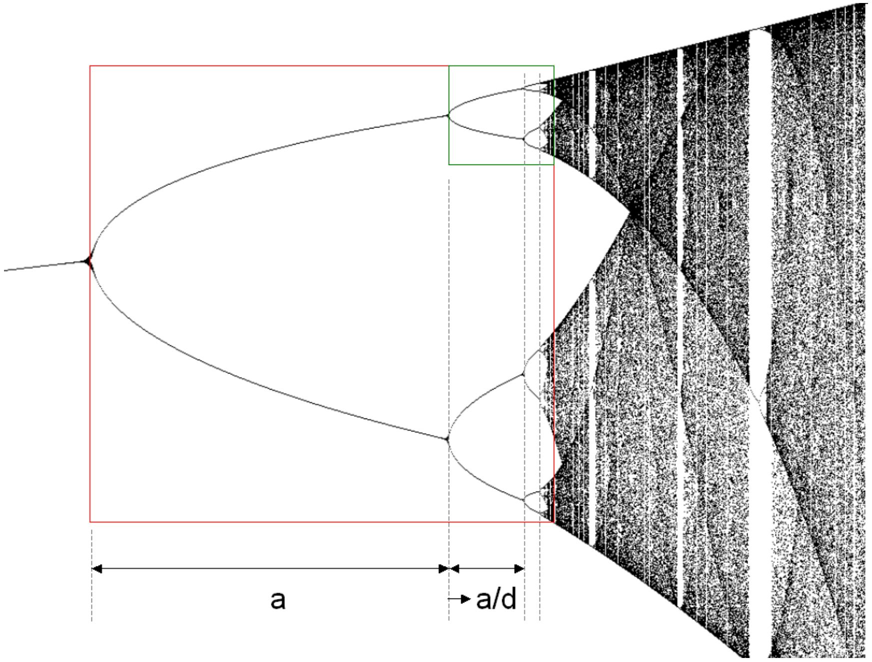
Although the Mandelbrot set is self-similar at magnified scales, the small scale details are not *identical* to the whole. In fact, the Mandelbrot set is infinitely complex.

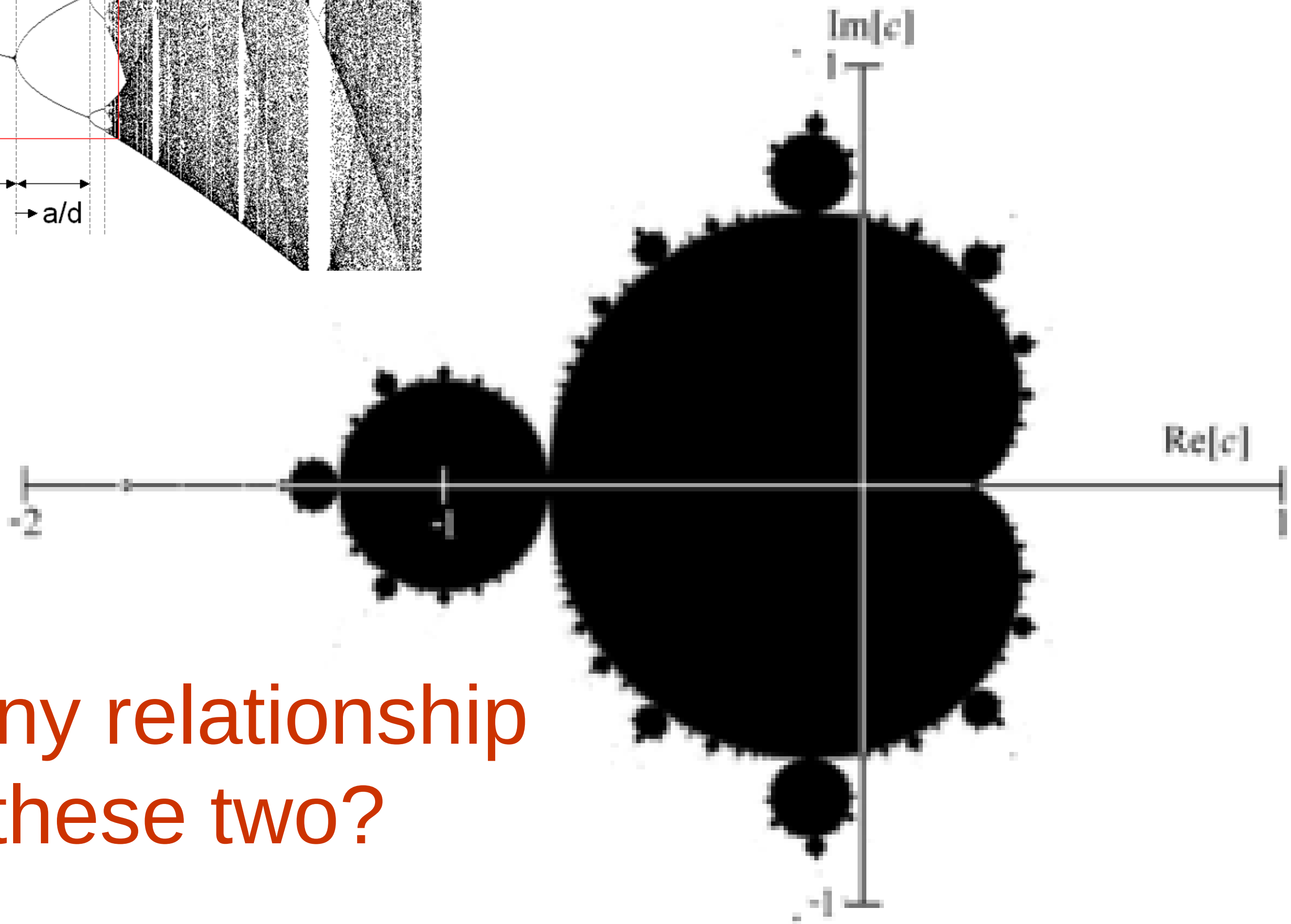
The process of generating the Mandelbrot set is simple, based on the simple equation involving complex numbers.


Im(z)
1
cardioid
bulb
-0.75,0.1
Re(z)
-2
-1
0.25
1
-1
r = a(1 − cos θ)
Sea-horse tail

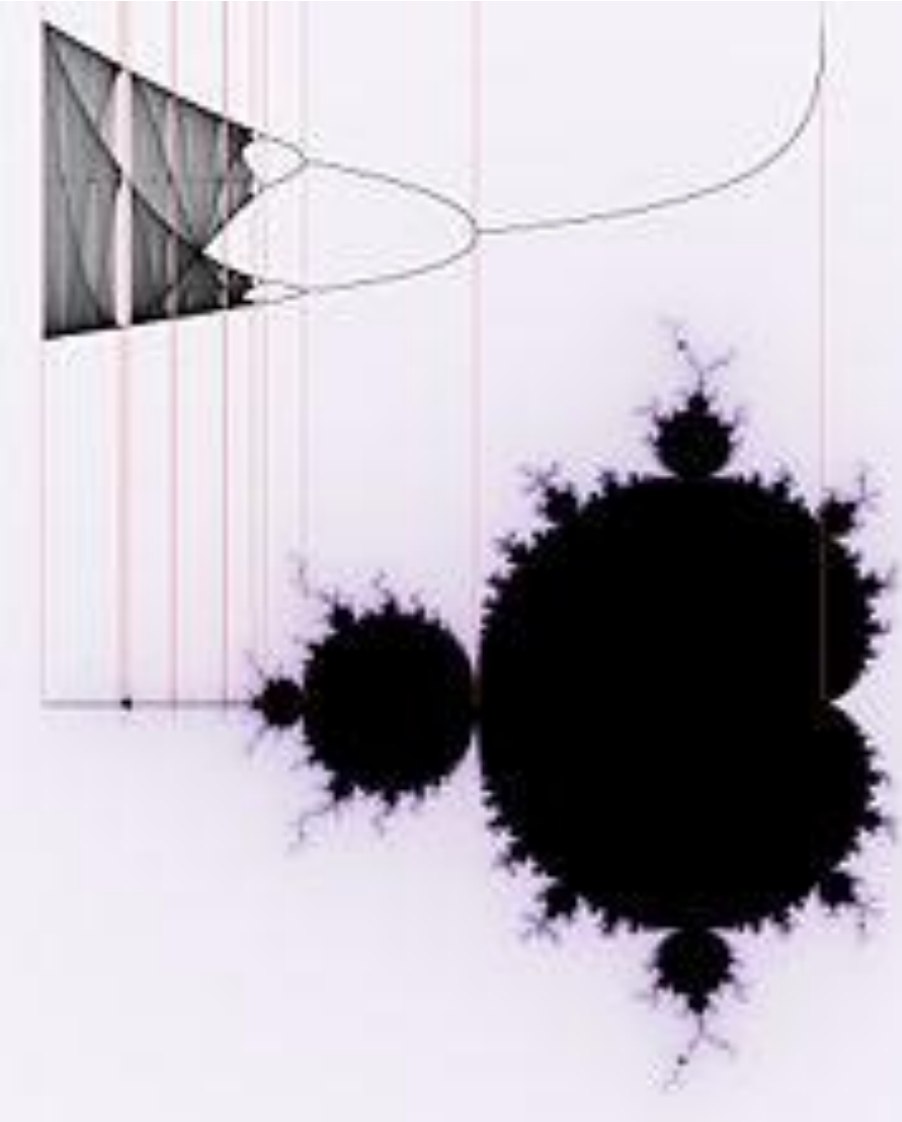
Logistic Map

Mandelbrot Set



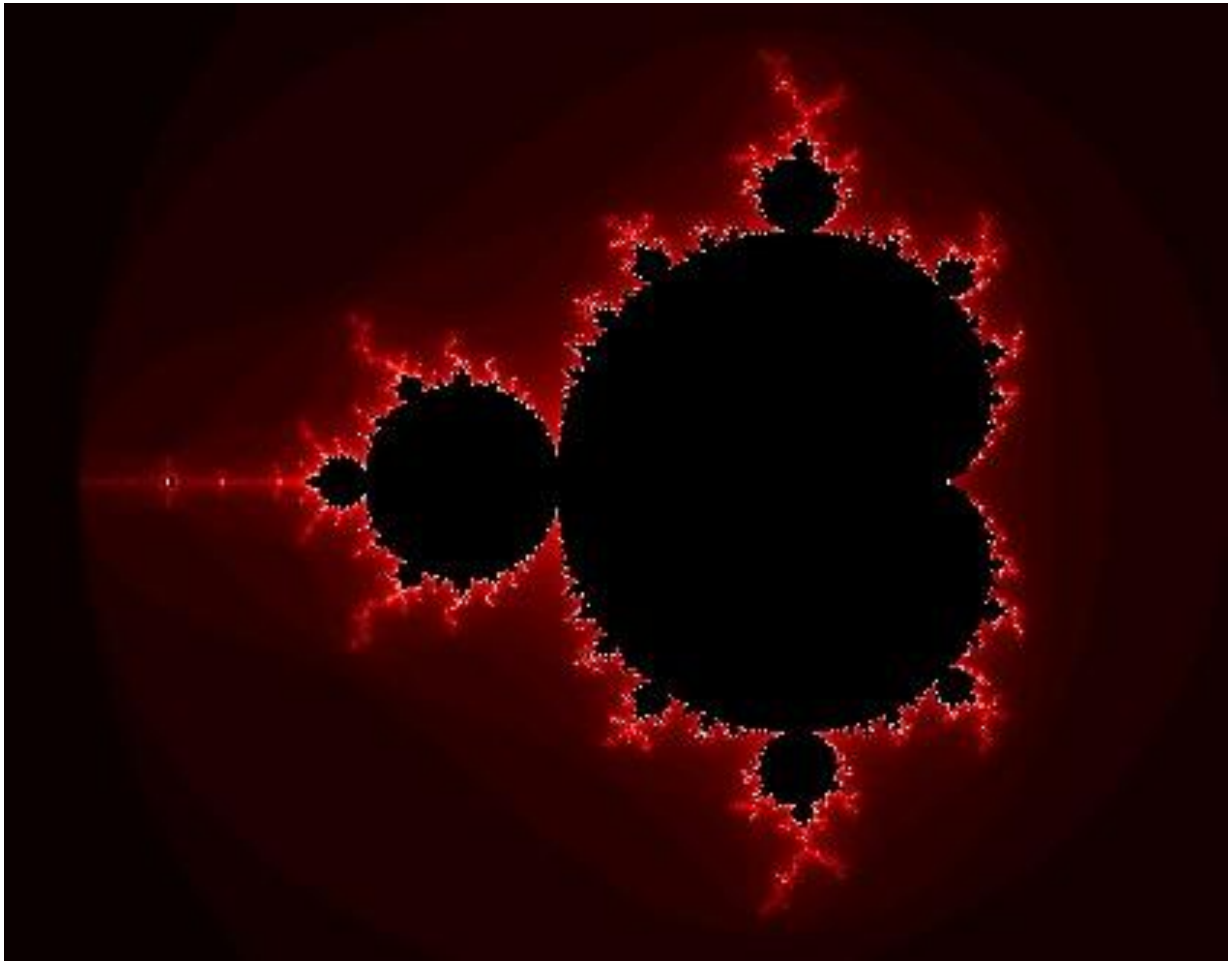


Is there any relationship between these two?



Courtsey: Public domain image by Georg-Johann Lay
http://en.wikipedia.org/wiki/File:Verhulst-Mandelbrot-Bifurcation.jpg#file

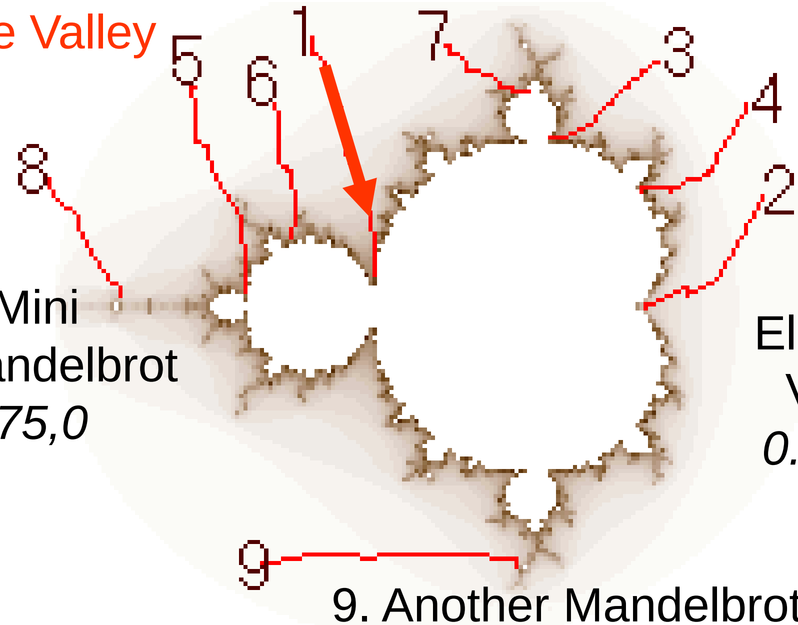
The first thing to do to draw the Mandelbrot set is to set the equivalence between pixel coordinates and complex numbers.



The colors in the images are shown in regions OUTSIDE the Mandelbrot set; the colors are chosen so that they have a mathematical relationship with C and the iterative mathematics.

SELF-SIMILARITY

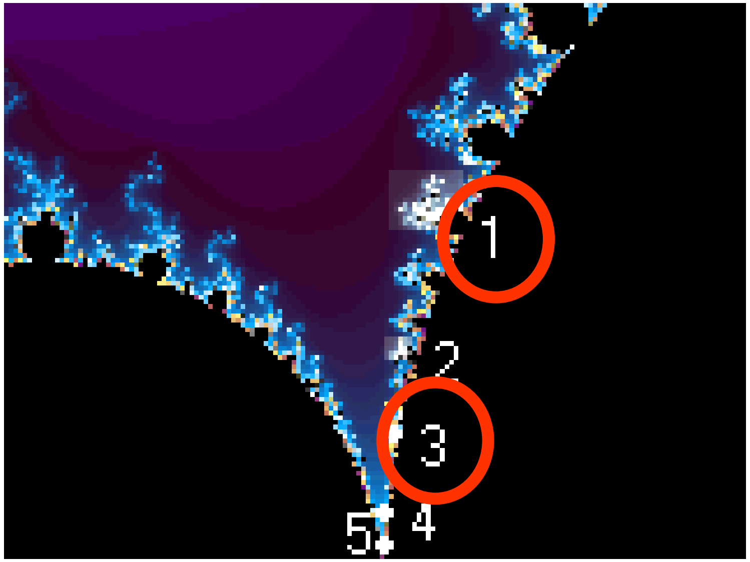
1.Seahorse Valley
-0.75, 0.1



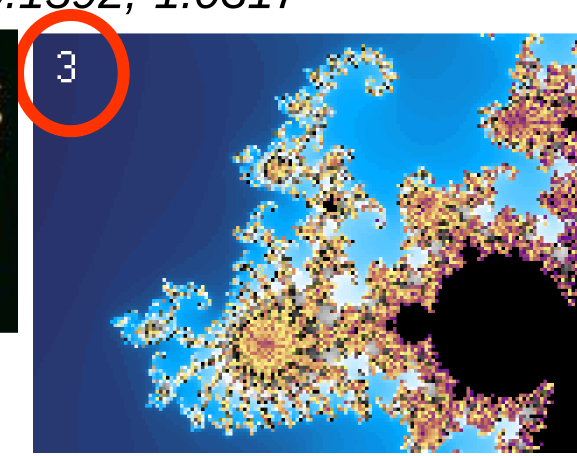
8. Mini Mandelbrot
-1.75,0

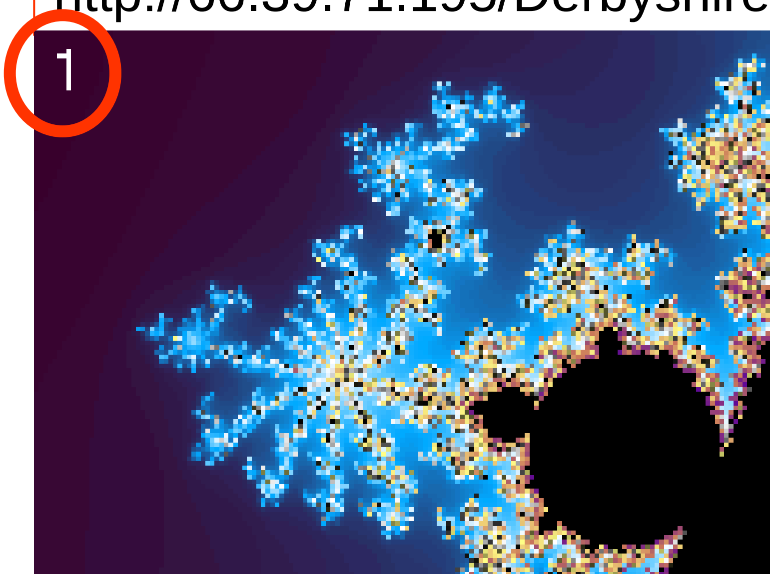
2 Elephant Valley
0.275, 0

9. Another Mandelbrot
-0.1592,-1.0317



http://66.39.71.195/Derbyshire/manguide.html





Mandelbrot Set Zoom on youtube ………………….Very Many!
For example:
(Jonathan Coulton's song on Mandelbrot song)
http://www.youtube.com/watch?v=gEw8xpb1aRA

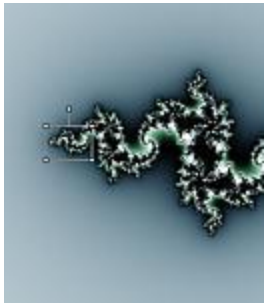
Some properties of the Mandelbrot set

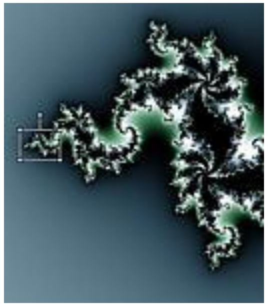
- M is connected; no disconnected "islands".

- Area of M: finite
  - it fits inside a circle of radius 2;
    the exact area has been approximated,
    but the length of its border is infinite.

- If you take any part of the border of the set, the length of this part will also be infinite. The border has **"infinite details".**

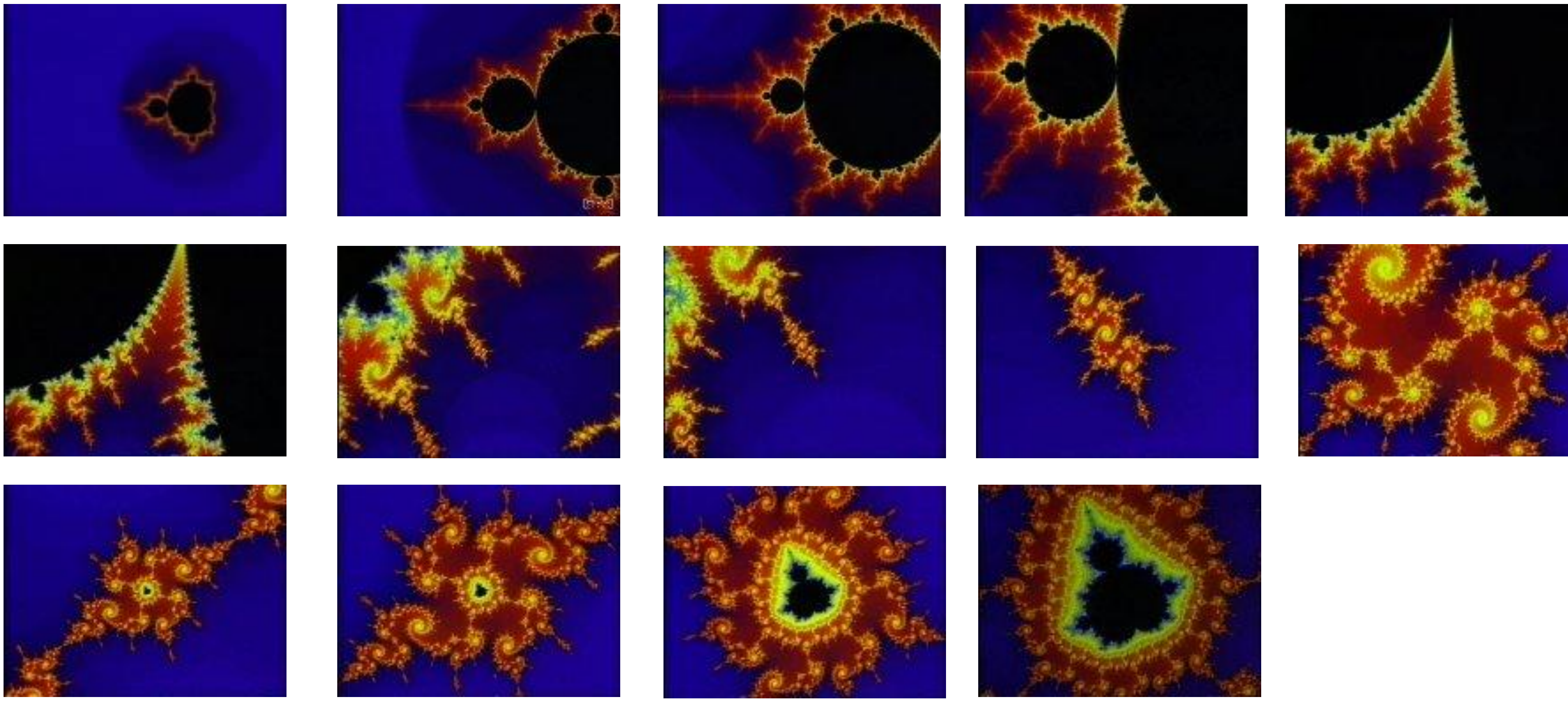
Fractal structures: Blood vessels branching out further and further, the branches of a tree, the internal structure of the lungs, graphs of stock market data, …..all have something in common: they are all self-similar.



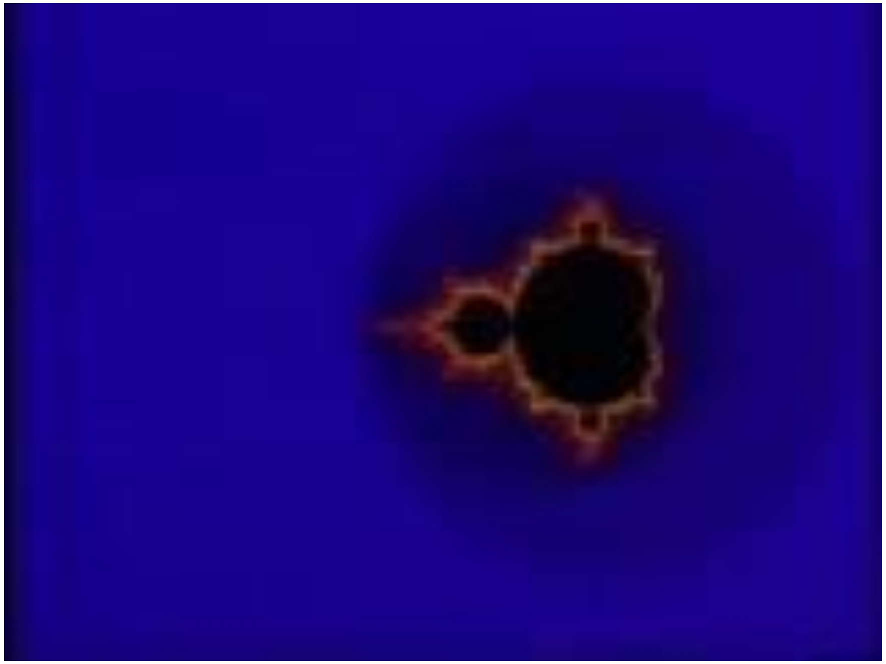
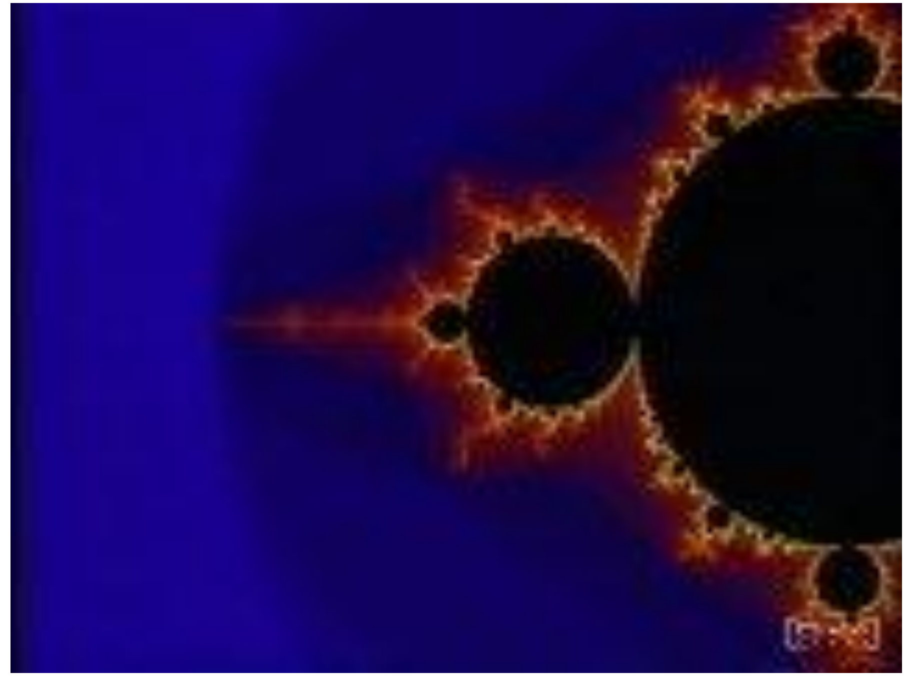
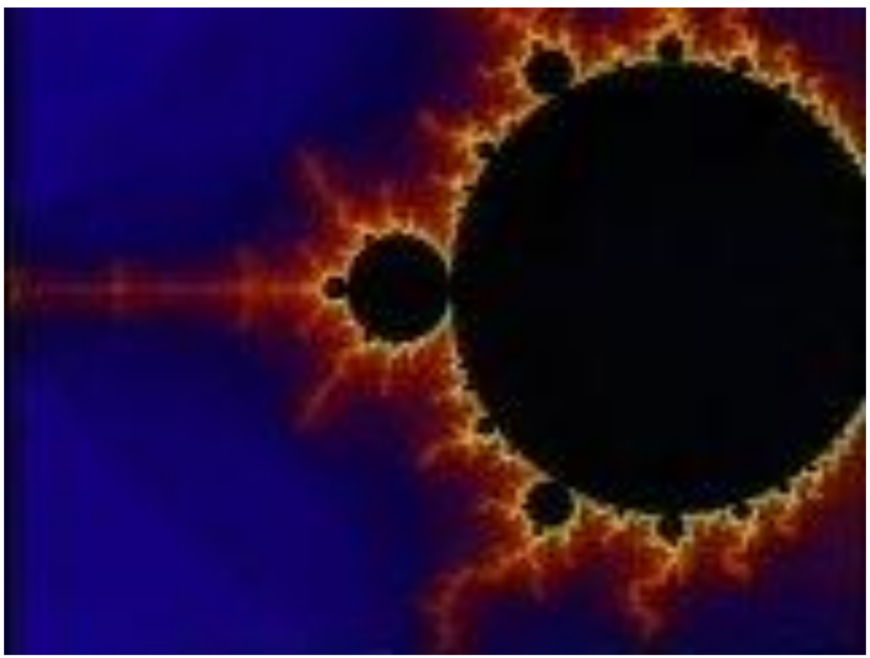
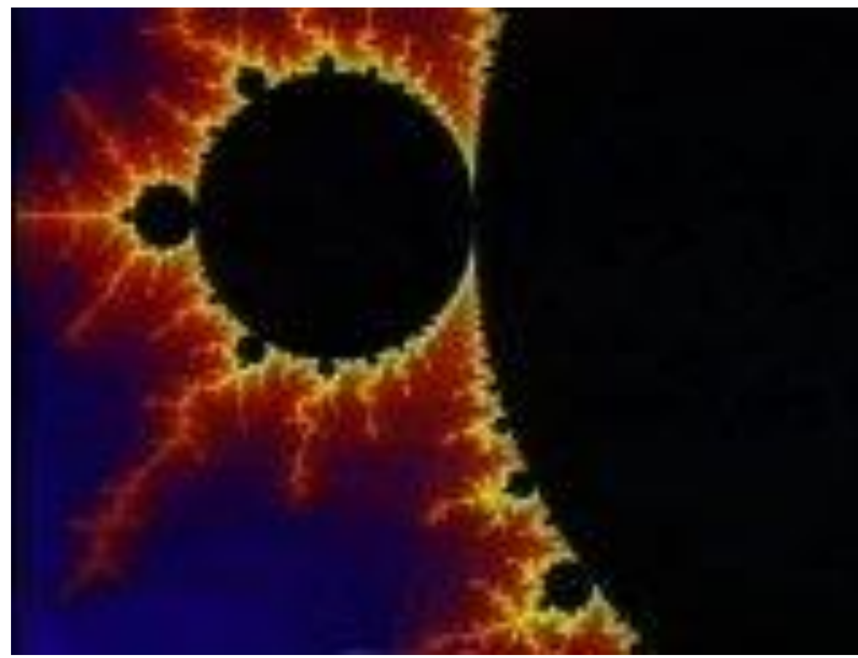


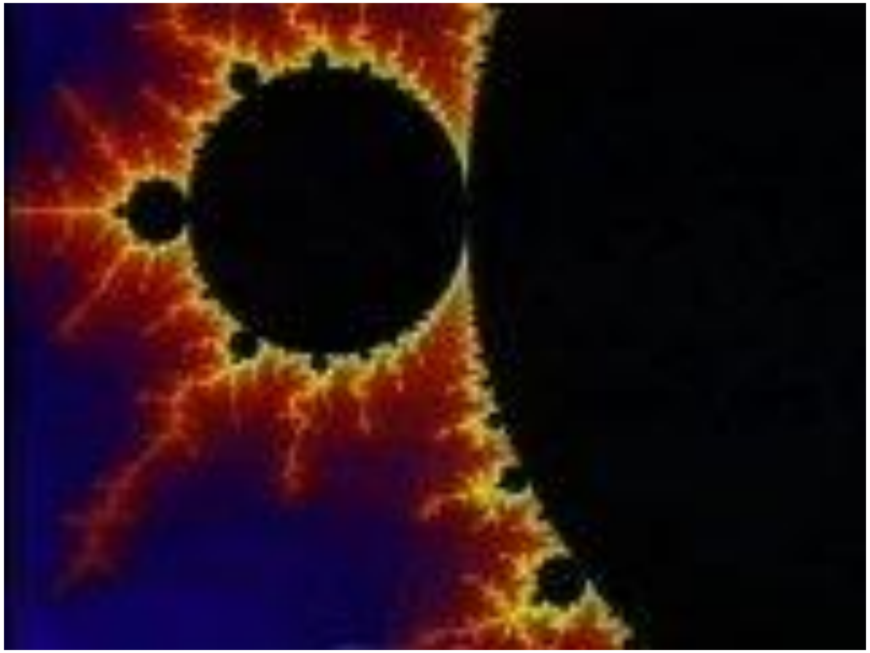
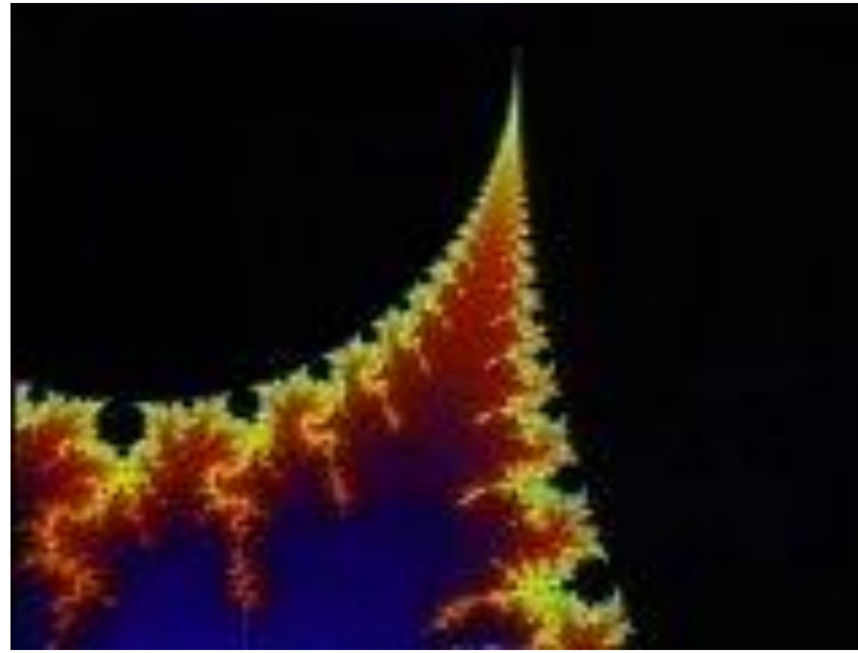
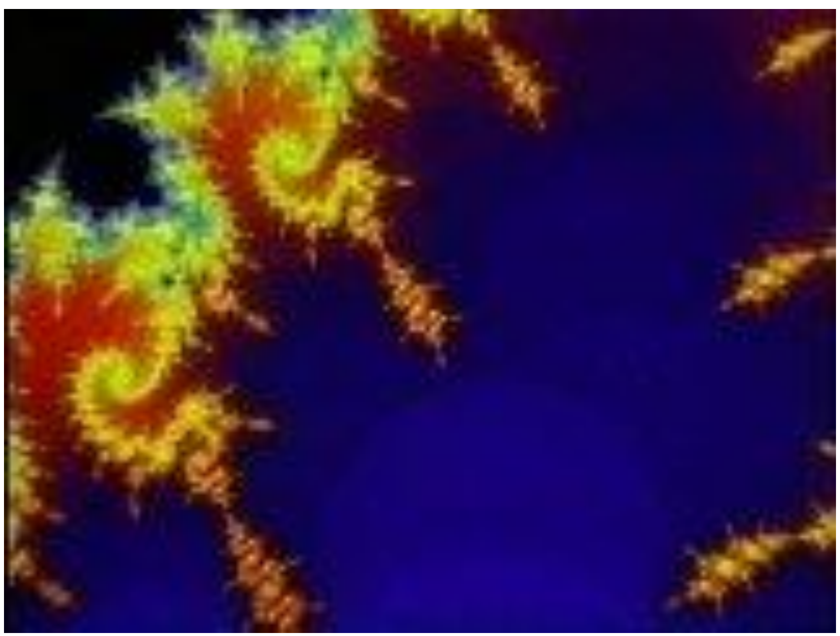
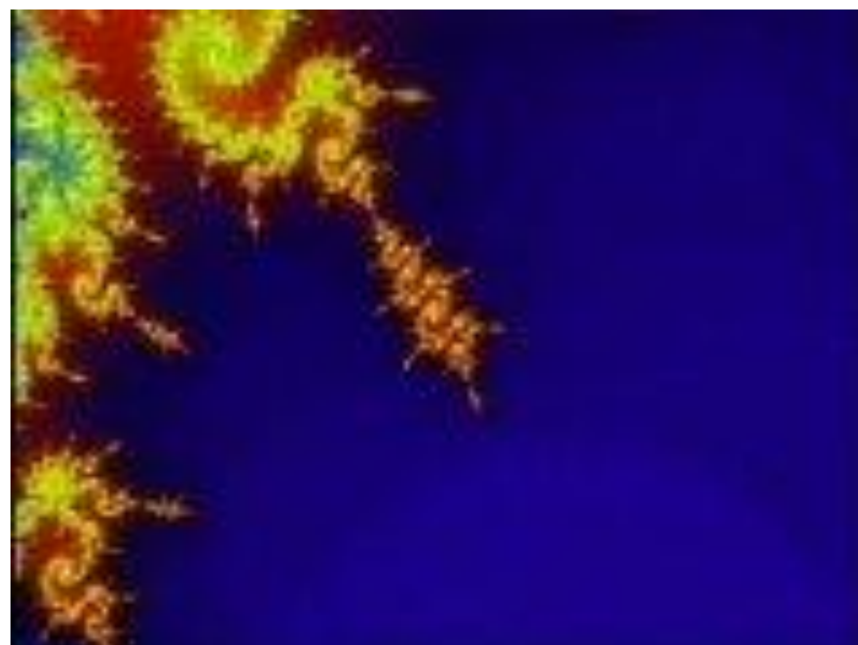
https://www.fractalus.com/info/layman.htm

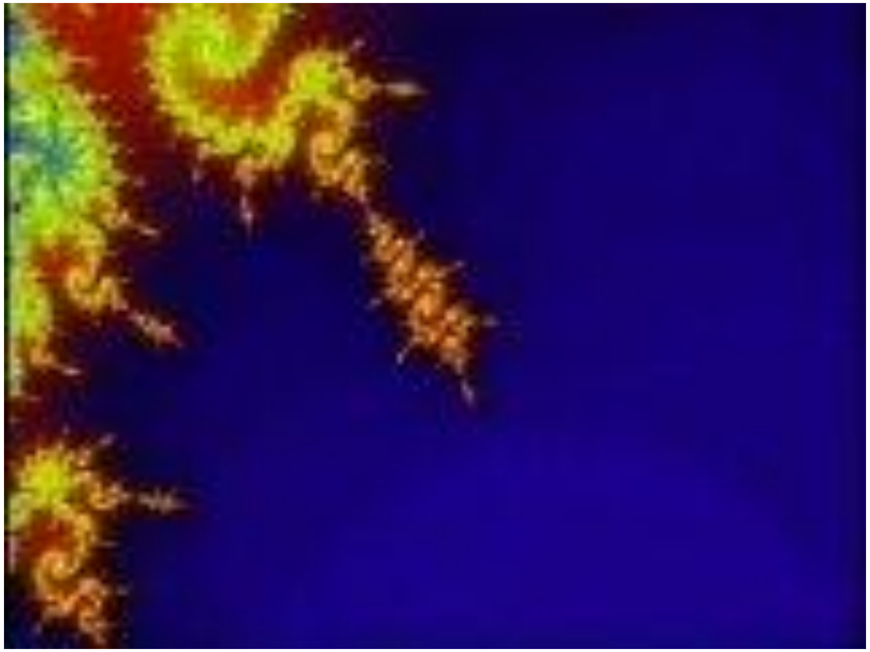
Conclude by showing video:
http://video.google.com/videoplay?docid=6460130356432628677#

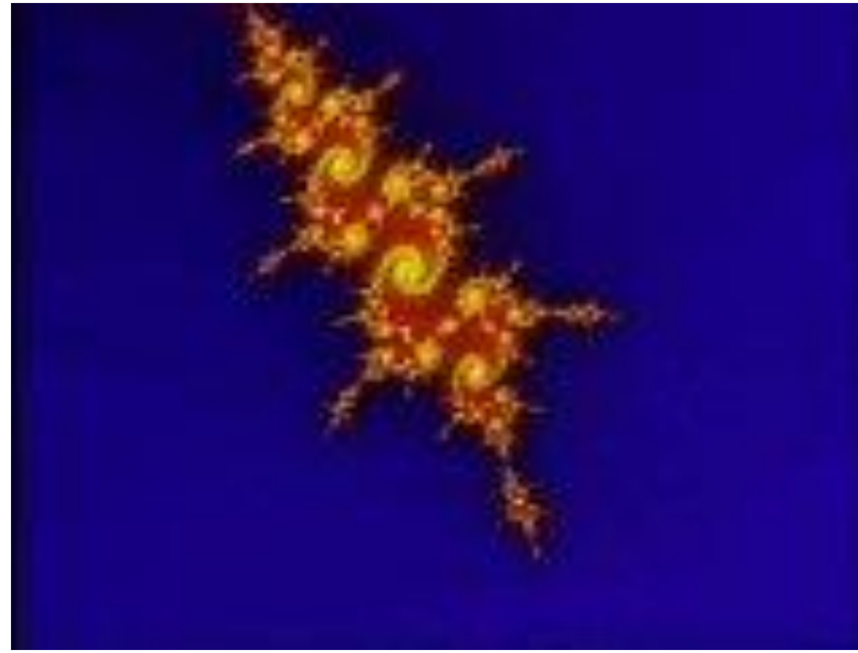


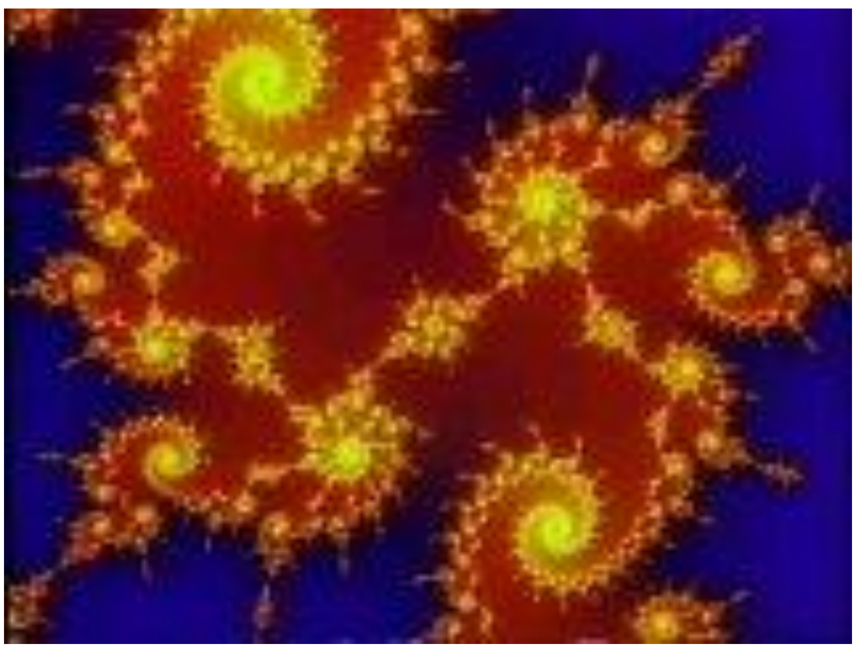
John Hubbard's video
The Beauty and Complexity of the Mandelbrot Set
which can be purchased on DVD via http://www.customflix.com/221873

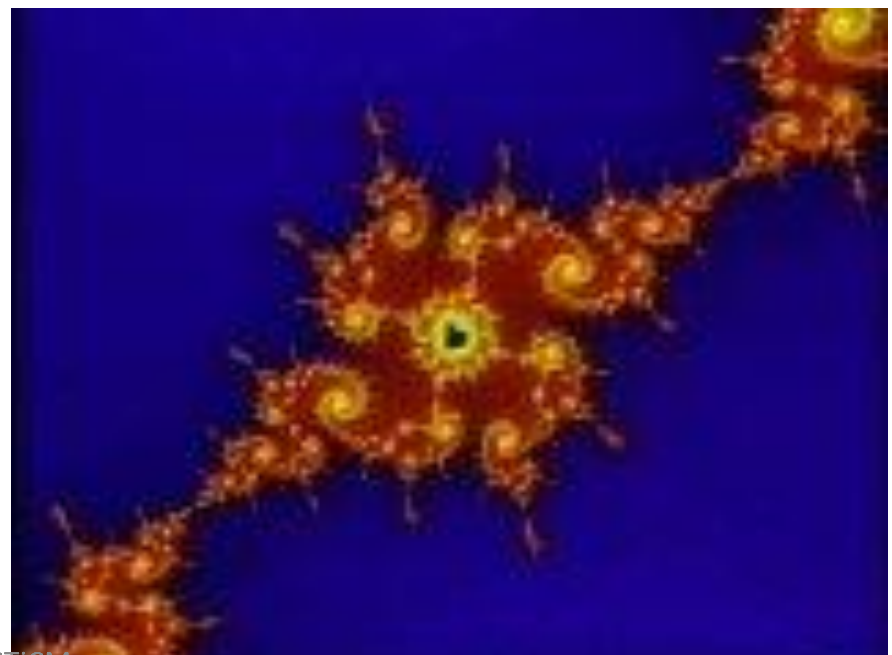







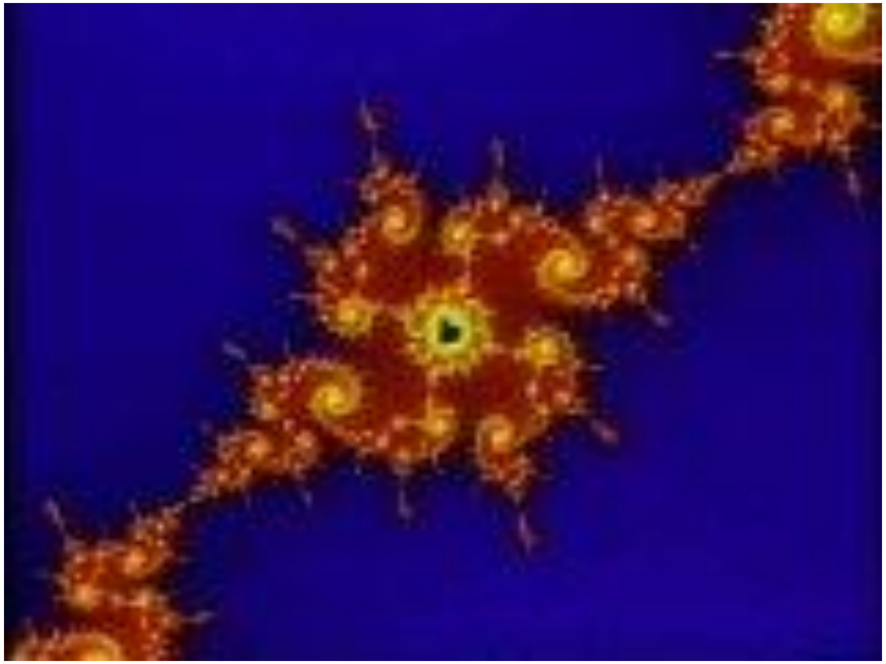
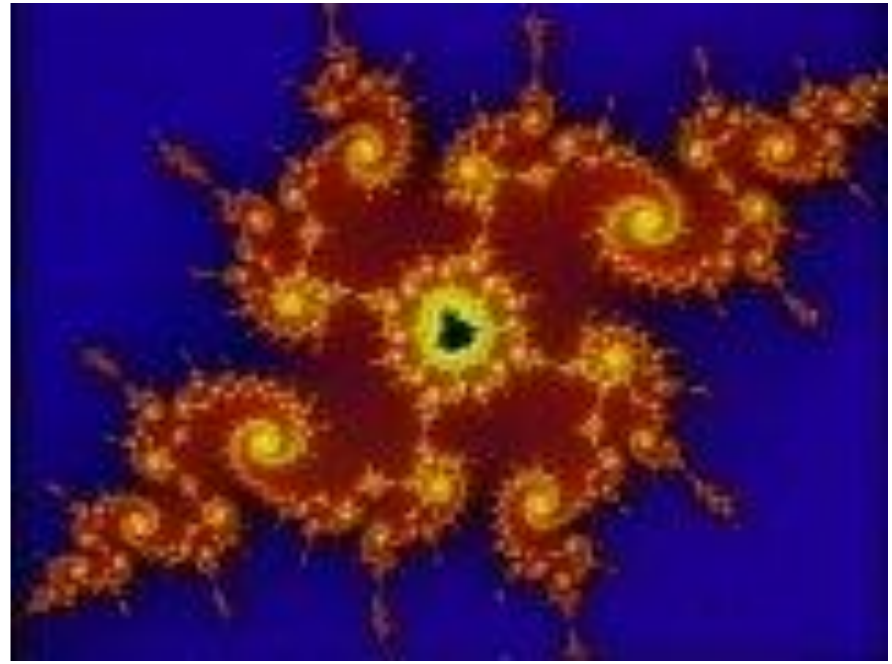
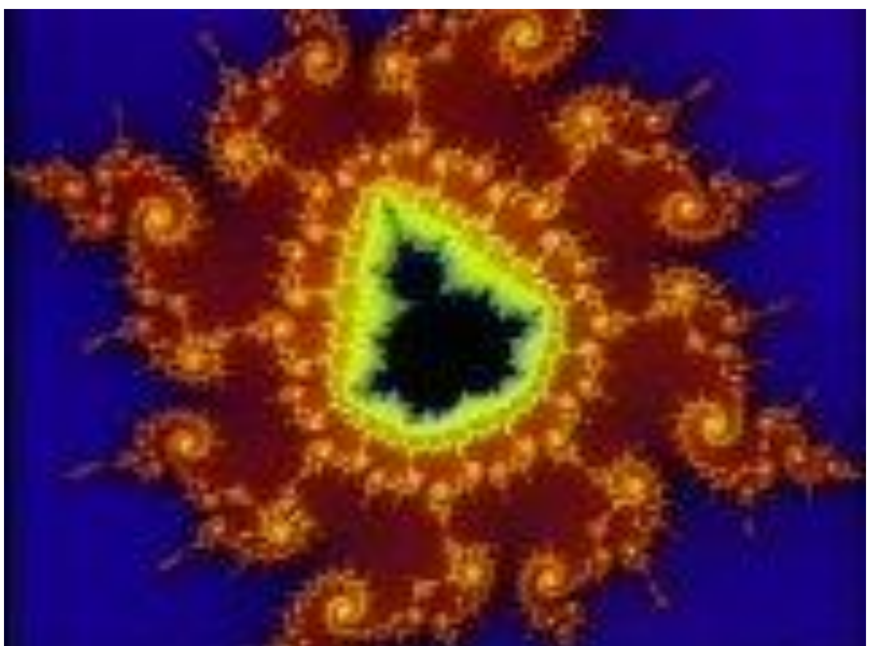
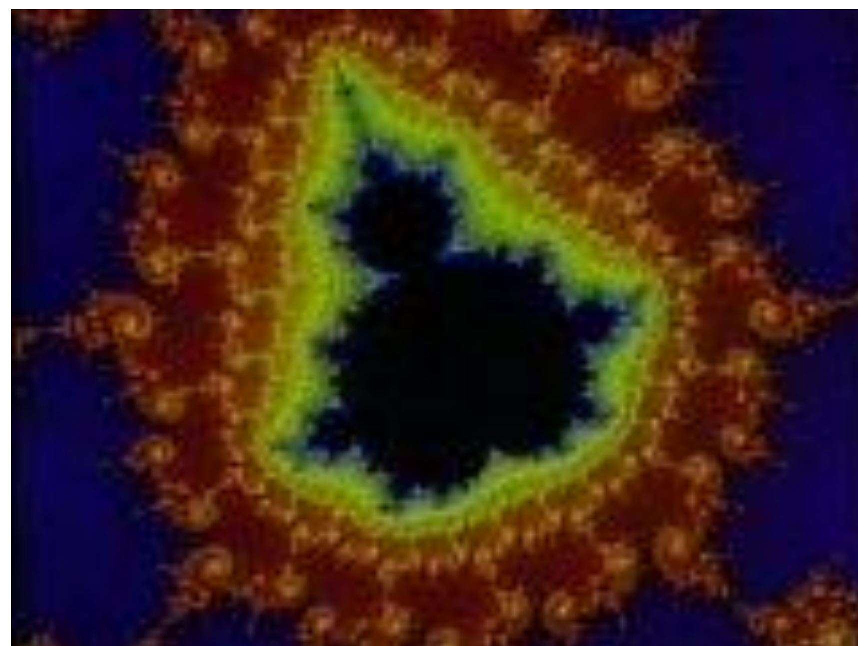






http://www.youtube.com/watch?v=gEw8xpb1aRA

Mandelbrot-Zoom-Carr-song.flv

References:

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Univ. College of London (1993)

Robert L, Devaney: A first course in
CHAOTIC DYNAMICAL SYSTEMS
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H.-O.Peitgen and P.H.Richter: The Beauty of Fractals
Springer-Verlag (1986)

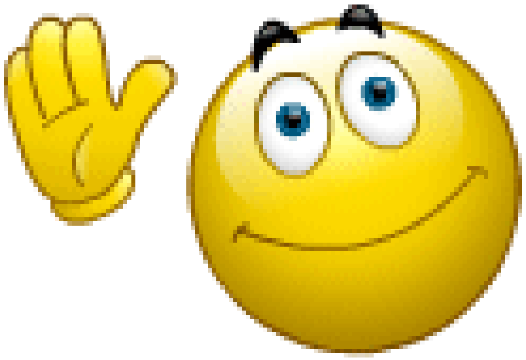
INTERNET ! Great source, but use it cautiously!!

We shall take a break here…….

Questions ? Comments ?

pcd@physics.iitm.ac.in http://www.physics.iitm.ac.in/~labs/amp/

pcdeshmukh@iitmandi.ac.in



Next: L40

Scope, and limitations of “Classical” Mechanics?